Semitopological coproducts and free objects on N totally ordered sets in some categories of complete, distributive, modular, and algebraic lattices

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Abstract

We characterize coproducts and free objects on indexed families of totally ordered sets in several categories of complete, distributive, modular and algebraic lattices via semitopologies on a product poset. Consequently, we derive necessary and sufficient conditions for the factorization of N distinct filtrations of a complete modular algebraic lattice through a completely distributive lattice $\mathbf{L}$. In the case $N = 2$, this condition realizes a simple compatibility criterion. Various incidental results are reported, which include a special role for well orders.

1 Introduction

Multifiltrations of modular lattices have recently begun a more active role in topological data analysis [5, 6, 7]. By definition, a filtration on an object A in an abelian category is an order preserving map $\mathbf{I} \to \mathrm{Sub}(A)$, where $\mathbf{I}$ is a totally ordered set and $\mathrm{Sub}(A)$ is the lattice of subobjects of A, ordered under inclusion. As such, it is natural to study universal objects – including coproducts and free objects – associated to indexed families of lattice homomorphisms $(f^\alpha : \mathbf{I} \to \mathbf{L})_{\alpha \in A}$ in categories common to homological algebra. These including the following.

- CD the category of complete, completely distributive lattices and complete lattice homomorphisms
- UD the category of upper continuous distributive lattices and lattice homomorphisms
- MA the category of modular algebraic lattices and complete lattice homomorphisms
- BM the category of bounded modular lattices and bound preserving lattice homomorphisms

BD the category of bounded distributive lattices and bound preserving lattice homomorphisms

Main results are stated and proved in §2. In §3 we supply supporting technical arguments. In §4 we prove Theorem 10, the main technical result.

1.1 Motivation and background

The free modular lattice on two chains plays a fundamental role in the homological algebra generally, and Puppe exact categories in particular [3]. It has great practical utility in studying pairs of filtrations $f^0, f^1 : \mathbf{I} \to \mathrm{Sub}(A)$, where $\mathbf{I}$ is a finite totally ordered set, A is an object in an abelian category, and $\mathrm{Sub}(A)$ is the lattice of subobjects of A. Such *bifiltrations* are common in homological algebra, arising, for example, from every nested family of topological spaces $(X_i)_{i\in\mathbf{I}}$.

Difficulties arise when $\mathbf{I}$ is not finite. Much of the machinery developed in homological algebra subtly relies upon the fact that the free modular lattice on two chains is a complete lattice and, more specifically, isomorphic to an Alexandrov topology. The free modular lattice on two infinite chains fails this condition, as does the free complete modular lattice. The former is not complete, and if the latter could be realized as an Alexandrov topology then every complete modular lattice would easily be shown to be upper and lower continuous, hence completely distributive.

In fact, CD the correct category in which to work if one demands that the free object be an Alexandrov topology. Its free objects (with respect to the forgetful functor to the category of partially ordered sets and monotone maps between them) *are* Alexandrov topologies [11], and if a pair of filtrations factor through any complete homomorphism $\mathbf{K} \to \mathbf{L}$ with $\mathbf{K}$ an Alexandrov topology, then their image necessarily lies within a completely distributive sublattice lattice, and a complete map from the free CD lattice is available.

The principle question, therefore, is not what free object to choose, but how to determine whether a pair of totally ordered subsets of a complete modular lattice extend to a completely distributive sublattice of $\mathbf{L}$. An exact criterion, in the special case where $\mathbf{L}$ is algebraic, is among the principle contributions of this text.

Theorem 12. *If $\mathbf{I}^1$, $\mathbf{I}^2$ are complete totally ordered sublattices of a modular algebraic lattice $\mathbf{L}$, then $\mathbf{I}^1 \cup \mathbf{I}^2$ extends to a complete, completely distributive sublattice iff*

$$\bigwedge_{b\in B}(a \vee b) = a \vee \bigwedge_{b\in B} b$$

for each element $a \in \mathbf{I}^1 \cup \mathbf{I}^2$ and each set B contained in $\mathbf{I}^1$ or in $\mathbf{I}^2$.

This result affords a substantial extension to the existing theory of homological persistence. In [7], for example, we remove the constructibility criterion imposed by Patel to define the type-B persistence diagram [9], and evince a novel stability result.

While single parameter persistence plays a central role in many modern applications, multiparameter persistence has matured as a field in its own right. The ideas of this text naturally extend to N-parameter persistence, and its companion work [7] has stimulated novel stability results in multiparameter generalized persistence [8]. Where possible, we therefore state results in terms filtrations indexed over arbitrary sets, rather than restricting to bifiltrations.

1.2 Notation and conventions

By *complete* lattice we mean a lattice with arbitrary meets and joins. An *unbounded complete* lattice is one with arbitrary nonempty meets and joins. A *complete* lattice homomorphism is a map of complete lattices that preserves arbitrary meets and joins. An *unbounded complete* lattice homomorphism is a map of unbounded complete lattices that preserves arbitrary nonempty meets and joins.

Given a partially ordered set P, we write P^* for the order dual. A poset is *bounded* if it has top and bottom elements, denoted 1_P and 0_P, respectively, or, where context leaves no room for confusion, simply 1 and 0. A poset homomorphism $f : P \to Q$ is bounded if both P and Q are bounded, and f preserves bounds. A (bounded) filtration of P is a (bounded) poset homomorphism $\mathbf{I} \to P$, where $\mathbf{I}$ is a totally ordered set. If P is a sub-poset of a Q and $q \in Q$, we write $P_{\leq q}$ for the set $\{p \in P : p \leq q\}$. For convenience, given $p \in P$ put $\mathring{\downarrow}(p) = \{q \in P : q < p\}$.

An element j of a lattice $\mathbf{L}$ is *completely join irreducible* if $\bigvee S = j$ implies $j \in S$. In this case the strict lower bounds of j contain a unique greatest element, denoted $\mathrm{pred}(j)$. A set $S \subseteq \mathbf{L}$ is *join-dense* if every element of $\mathbf{L}$ can be expressed as a join of some (possibly empty, possibly infinite) subset of S. Meet density is dual. We say that $\mathbf{L}$ has completely dense irreducibles if completely join irreducibles are join dense and completely meet irreducibles are meet dense. We denote

- $\mathbb{J}(\mathbf{L})$ the poset of completely join irreducible elements of $\mathbf{L}$
- $\mathbb{P}(P)$ the lattice of subsets of P, ordered under inclusion
- $\mathbb{A}(P)$ the lattice of decreasing subsets of P, ordered under inclusion; equivalently, the Alexandrov topology of P^*
- $\mathbb{S}(P)$ the bounded sublattice of $\mathbb{A}(P)$ generated by all sets of form $\downarrow(p)$

We similarly write $\mathbb{J}^*(\mathbf{L})$ for the set of completely meet irreducible elements of $\mathbf{L}$, ordered as in $\mathbf{L}$. Note that lattice $\mathbb{S}(P)$ is a *semitopology* on the ground set of P. However, it is not an Alexandrov topology, complete lattice, or complete sublattice of $\mathbb{A}(P)$, in general.

2 Main results

The following results are among the most relevant to current applications in topological data analysis. We group them according to universal property:

map extensions, free objects, and coproducts. Top-level proofs appear together with their theorems. Supporting results may be found in §3.

2.1 Extension maps

Theorem 1 (Extensions to CD, Tunnicliffe 1974, [10])**.** *Suppose that a complete lattice* $\mathbf{K}$ *is generated by its subset* X*, and that* $\mathbf{L}$ *is completely distributive. Then a function* $f : X \to \mathbf{L}$ *extends to a complete homomorphism* $g : \mathbf{K} \to \mathbf{L}$ *if and only if* $\bigwedge f(S) \leq \bigvee f(T)$ *for all* $S, T \subseteq X$ *such that* $\bigwedge S \leq \bigvee T$*. The extension is unique, if it exists.*

Lemma 2. *Let* f *be an arbitrary, possibly unbounded homomorphism from a distributive lattice* $\mathbf{K}$ *to a complete lattice* $\mathbf{L}$*. Suppose that* $\mathbf{K}$ *has completely dense irreducibles, and assume that every element of* $\mathbf{K}$ *may be expressed as a finite join of elements of* S*, for some fixed* $S \subseteq \mathbf{K}$*. If* $\mathbf{L}$ *is complete, then* f *preserves existing (finite, empty, and infinite) joins iff* $f(s) = \bigvee_{j \prec s} f_j$ *for all* $s \in S$*.*

Proof. Let T be any subset of $\mathbf{K}$ such that $\bigvee T$ exists. By hypothesis, there exists finite $U \subseteq S$ such that $\bigvee T = \bigvee U$. Proposition 17 implies that $\bigcup_{t \in T} \mathbb{J}(\mathbf{K})_{\leq t} = \bigcup_{u \in U} \mathbb{J}(\mathbf{K})_{\leq u}$, hence $j \prec u$ for some $u \in U$ iff $j \prec t$ for some $t \in T$. Consequently,

$$f_{\bigvee T} = \bigvee_{u \in U} f_u = \bigvee_{u \in U} \bigvee_{j \prec u} f_j \leq \bigvee_{t \in T} f_t.$$

The reverse inequality is clear, since $f_{\bigvee T}$ bounds every f_t from above. $\square$

2.2 Free objects

First let us recall some classical results on free objects in CD and BM. Given a poset P, write $\hat{\mathbb{A}}(P)$ for the family of decreasing subsets $S \subseteq P$ such that both S and $P - S$ are nonempty.

Definition 1. *If* P *is a partially ordered set and* $\mathbb{X} \in \{\mathbb{A}, \hat{\mathbb{A}}\}$*, then the* free embedding $\mu : P \to \mathbb{A}\mathbb{X}(P)$ *is the map defined by* $\mu(p) = \{T \in \mathbb{X}(P) : p \notin T\}$ *for all* $p \in P$*.*

Remark 1. *Let* $\mu : P \to \mathbb{A}\mathbb{X}(P)$ *be the free embedding.*

1. *If* $\mathbb{X} = \mathbb{A}$ *then* $\mu(p) = \mathbb{A}(P - \uparrow p)$*.*
2. *If* $\mathbb{X} = \mathbb{A}$ *then* μ *fails to preserve existing top and bottom elements.*
3. *If* $\mathbb{X} = \hat{\mathbb{A}}$ *then* μ *preserves existing top and bottom elements.*

Theorem 3 (Tunnicliffe 1985, [11])**.** *Let* P *be a partially ordered set,* $\mathbf{L}$ *be a complete, completely distributive lattice, and* $f : P \to \mathbf{L}$ *be an an order-preserving function. In the following diagram,*

$$\begin{tikzcd} & \mathbb{A}\mathbb{X}(P) \arrow[dr, "g"] & \\ P \arrow[ur, "\mu"] \arrow[rr, "f"] & & \mathbf{L} \end{tikzcd}$$

let μ denote the free embedding.

1. *If $\mathbb{X} = \mathbb{A}$ then the diagram commutes for exactly one complete lattice homomorphism g.*
2. *If $\mathbb{X} = \hat{\mathbb{A}}$ then the diagram commutes for exactly one unbounded-complete lattice homomorphism g.*

In either case, the unique commuting homomorphism satisfies

$$g(S) = \bigvee_{X \in M_S} \bigwedge f(X) = \bigwedge_{Y \in N_S} \bigvee f(Y)$$

where $M_S = \{X \subseteq P : \bigwedge \mu(X) \subseteq S\}$ and $N_S = \{X \subseteq P : S \subseteq \bigvee \mu(X)\}$.

Corollary 4 (Free objects in CD, Tunnicliffe 1985, [11])**.** *Let* BP *denote the category of bounded posets and bounded poset homomorphisms. Likewise let* $\hat{\mathrm{B}}\mathrm{P}$ *denote the category of posets and poset homomorphisms, and* $\hat{\mathrm{C}}\mathrm{D}$ *denote the category of unbounded-complete, completely distributive lattices and unbounded-complete homomorphisms.*

1. *The forgetful functor* $\hat{\mathrm{C}}\mathrm{D} \to \hat{\mathrm{B}}\mathrm{P}$ *has a left adjoint carrying P to $\mathbb{A}\hat{\mathbb{A}}(P)$.*
2. *The forgetful functor* $\mathrm{CD} \to \mathrm{BP}$ *has a left adjoint carrying P to $\mathbb{A}\hat{\mathbb{A}}(P)$.*
3. *The forgetful functor* $\mathrm{CD} \to \hat{\mathrm{B}}\mathrm{P}$ *has a left adjoint carrying P to $\mathbb{A}\mathbb{A}(P)$.*

Definition 2. *Let $(P_\alpha)_{\alpha \in A}$ be a family of small partially ordered sets, and Q be any subset of the cartesian product $\prod_{\alpha \in A} P_\alpha$. The* canonical filtration of Q by P_α *is*

$$\lambda^\alpha : P_\alpha \to \mathbb{A}(Q) \qquad\qquad x \mapsto \{q \in Q : q_\alpha \leq x\}.$$

Similarly, the canonical filtration of Q by $\mathbb{A}(P_\alpha)$ *is the map*

$$\lambda^\alpha : \mathbb{A}(P_\alpha) \to \mathbb{A}(Q) \qquad\qquad S \mapsto \{q \in Q : q_\alpha \in S\}.$$

When these maps coincide with coprojections of a coproduct structure on $\mathbb{A}(Q)$, we call them canonical coprojections*.*

Given a sequence $\mathbf{I}^1, \ldots, \mathbf{I}^N$ of disjoint bounded chains, write $P(\mathbf{I}^1, \ldots, \mathbf{I}^N)$ for the partially ordered set obtained by identifying the top (respectively, the bottom) elements in each ground set.

Theorem 5 (Free objects in BD)**.** *Let $\mathbf{I}^1, \ldots, \mathbf{I}^N$ be disjoint bounded totally ordered sets. Then $\mathbb{S}(\prod_m \mathbf{I}^m_{>0})$ is the free object on $P(\mathbf{I}^1, \cdots, \mathbf{I}^N)$ with respect to the forgetful functor* $\mathrm{BD} \to \mathrm{BP}$*. Equivalently, it is a coproduct in* BD *with canonical coprojections $\lambda^m : \mathbf{I}^m \to \mathbb{S}(\prod_m \mathbf{I}^m_{>0})$.*

Proof. The proof is a rudimentary exercise of universal algebra, and closely mirrors that of Theorem 6. The main observation is that every element in the bounded sublattice generated by a collection of bounded totally ordered sublatices $C^1, \ldots, C^N$ can be expressed as a finite join of elements in the set $\{\bigwedge i : i \in \prod C\}$. This case is somewhat easier than Theorem 6, however, since distributivity comes for free, and we need no theorem of Birkhoff. □

Theorem 6 (Free objects in BM, Grandis [3]). *Let* $\mathbf{I}^1$ *and* $\mathbf{I}^2$ *be disjoint bounded totally ordered sets. Then* $\mathbb{S}(\mathbf{I}^1_{>0} \times \mathbf{I}^2_{>0})$ *is the free object on* $P(\mathbf{I}^1, \mathbf{I}^2)$ *with respect to the forgetful functor* BM $\to$ BP*. Equivalently, the cospan of canonical filtrations* $\mathbf{I}^1 \longrightarrow \mathbb{S}(\mathbf{I}^1_{>0} \times \mathbf{I}^2_{>0}) \longleftarrow \mathbf{I}^2$ *is a coproduct in* BM.

Proof. Claims 1 and 2 are evidently equivalent. A similar formulation and argument may be found in [3]. For convenience, identify $\mathbb{S}(\mathbf{I}^1_{>0} \times \mathbf{I}^2_{>0})$ with the lattice $\mathbb{V}(\mathbf{I}^1_{>0} \times \mathbf{I}^2_{>0})$ of finite antichains in $\mathbf{I}^1_{>0} \times \mathbf{I}^2_{>0}$ via the mutually inverse lattice isomorphisms $S \mapsto \max(S)$ and $T \mapsto \downarrow T$.

Let $\mathbf{L}$ be a bounded modular lattice. Given bounded lattice homomorphisms $F : \mathbf{I}^1 \to \mathbf{L}$ and $G : \mathbf{I}^2 \to \mathbf{L}$, define $f : \mathbb{V}(\mathbf{I}^1_{>0} \times \mathbf{I}^2_{>0}) \to \mathbf{L}$ by

$$\{(i_0, j_0), \dots, (i_m, j_m)\} \mapsto (F_{i_0} \wedge G_{i_0}) \vee \cdots \vee (F_{i_m} \wedge G_{i_m}).$$

When $\mathbf{I}^1$ and $\mathbf{I}^2$ are finite, a theorem of Birkhoff [1, Ch. III.7, Theorem 9] provides that f is a bounded lattice homomorphism extending F and G. Moreover, any such map is uniquely determined by its values on elements of form $\{(i, j)\} = \{(i, 1)\} \wedge \{(1, j)\}$, hence by F and G. When $\mathbf{I}^1$ and $\mathbf{I}^2$ are infinite, every pair of finite antichains lies in a finite bounded sublattice of form $\mathbb{V}(\hat{\mathbf{I}}^1_{>0}, \hat{\mathbf{I}}^2_{>0})$ for some $\hat{\mathbf{I}}^m \subseteq \mathbf{I}^m$, so f preserves meets and joins. Uniqueness is clear. $\square$

2.3 Coproducts in CD

Theorem 7. *Suppose that* P *is an* A*-indexed family of nonempty posets. Let* $Q = \prod_A P$*, and let* $\lambda^\alpha : \mathbb{A}(P_\alpha) \to \mathbb{A}(Q)$ *be the canonical filtration of* $\mathbb{A}(Q)$ *by* $\mathbb{A}(P_\alpha)$*. Then the pair* $(\mathbb{A}(Q), \{\lambda^\alpha : \alpha \in A\})$ *is a coproduct of the indexed family* $(\mathbb{A}(P^\alpha))_{\alpha \in \mathbf{I}}$ *in the category of complete, completely distributive lattices and complete lattice homomorphisms.*

Proof. Put $X = \bigcup_\alpha \lambda^\alpha(P_\alpha)$, and let $S_\alpha = S \cap \lambda^\alpha(P_\alpha)$ for each $S \subseteq X$. Fix a complete, completely distributive lattice $\mathbf{L}$, and for each $\alpha \in A$ let $f_\alpha : \mathbb{A}(P_\alpha) \to \mathbf{L}$ be a complete lattice homomorphism. Since $\lambda^\alpha(P_\alpha) \cap \lambda^\beta(P_\beta) = \emptyset$ when $\alpha \neq \beta$, there is a unique map $f : X \to \mathbf{L}$ such that

$$f\lambda^\alpha = f_\alpha$$

for each $\alpha \in T$.

Posit $S, T \subseteq X$ such that $\bigwedge S \leq \bigvee T$. We claim there exists $\alpha \in A$ such that $\bigwedge S_\alpha \leq \bigvee T_\alpha$. Otherwise there exists, for each $\beta \in T$, an element $s_\beta \in (\bigwedge S_\beta) - (\bigvee T_\beta)$. Arranging these elements into an A indexed family yields $(s_\beta)_{\beta \in A} \in (\bigwedge S) - (\bigvee T)$, a contradiction. Thus

$$\bigwedge f(S) = \bigwedge_{\alpha \in A} \bigwedge f(S_\alpha) \leq \bigvee_{\alpha \in A} \bigvee f(T_\alpha) = \bigvee f(T).$$

Therefore f has a unique extension to $\mathbb{A}(Q)$, by Theorem 1. $\square$

When $\mathbf{J}$ is a complete totally ordered lattice with completely dense irreducibles the canonical filtration $\mathbf{J} \to \mathbb{A}(\mathbb{J}(\mathbf{J}))$ is a lattice isomorphism, which we will call the *canonical isomorphism*. Corollary 8 is the special case of Theorem 7 obtained by identifying $\mathbf{I}^\alpha$ with $\mathbb{A}(\mathbb{J}(\mathbf{J}^\alpha))$ under this isomorphism. We state this case explicitly as it holds special significance to the study of homological persistence.

Corollary 8 (Chains and dense irreducibles)**.** *Let* $\mathbf{I}$ *be an* A*-indexed family of complete totally ordered lattices with completely dense irreducibles, and let* $\alpha \in A$ *be given. Let* $Q = \prod_{\alpha \in A} \mathbb{J}(\mathbf{I}^\alpha)$ *and let* $\lambda^\alpha : \mathbf{I}^\alpha \to \mathbb{A}(Q)$ *be the canonical filtration of* $\mathbb{A}(Q)$ *by* $\mathbf{I}^\alpha$*. Then the pair* $(\mathbb{A}(Q), \{\lambda^\alpha : \alpha \in A\})$ *is a coproduct of the indexed family of totally ordered sets* $\mathbf{I}$ *in the category of complete, completely distributive lattices and complete lattice homomorphisms.*

Corollary 9 may be viewed as the special case of Corollary 8 where the totally ordered set $\mathbb{J}(\mathbf{I}^\alpha)$ has completely dense irreducibles, for each α. In this case all objects are free.

Corollary 9 (Free objects)**.** *Let* $\mathbf{I}$ *be an* A*-indexed family of complete totally ordered lattices with completely dense irreducibles. Then diagram* (1) *commutes for each* $\alpha \in A$*, where (i)* λ^α *is the canonical filtration* $\mathbb{A}(\mathbf{I}^\alpha) \to \mathbb{A}(\prod_{\alpha \in A} \mathbf{I}^\alpha)$*, (ii)* $\eta^\alpha(\mathbf{S}) = \{S \cup T : S \in \mathbf{S},\ T \subseteq \bigcup_{\beta \neq \alpha} \mathbb{J}(\mathbf{I}^\beta)$*, (iii)* $\mathrm{Pred}(j) := \downarrow \mathrm{pred}(j)$*, (iv)* ϕ^α *is the isomorphism induced by the canonical identification* $\mathbf{I}^\alpha \equiv \mathbb{A}(\mathbb{J}(\mathbf{I}^\alpha))$*, (iv)* ψ *is the isomorphism induced by the evident identification* $\mathbb{A}(\bigcup_{\alpha \in A} \mathbb{J}(\mathbf{I}^\alpha)) \equiv \prod_{\alpha \in A} \mathbb{A}(\mathbb{J}(\mathbf{I}^\alpha))$*, and (vi)* μ^0 *and* μ^1 *are the canonical injections described in Theorem 3.*

$$
\begin{tikzcd}
\mathbb{A}(\mathbf{I}^\alpha) \arrow[rr, "\lambda^\alpha"] \arrow[dd, "\phi^\alpha"', "\cong"] & & \mathbb{A}(\prod_{\alpha \in A} \mathbf{I}^\alpha) \arrow[dd, "\psi"', "\cong"] \\
& \mathbb{J}(\mathbf{I}^\alpha) \arrow[ul, "\mathrm{Pred}"'] \arrow[dl, "\mu^0"'] \arrow[dr, "\mu^1"] & \\
\mathbb{A}^2(\mathbb{J}(\mathbf{I}^\alpha)) \arrow[rr, "\eta^\alpha"] & & \mathbb{A}^2(\bigcup_{\alpha \in A} \mathbb{J}(\mathbf{I}^\alpha))
\end{tikzcd}
\tag{1}
$$

Consequently, (i) the objects in (1) *are free, specifically with respect to the forgetful functor to the category of posets and order preserving functions, and (ii) the pairs* $(\mathbb{A}(\prod_{\alpha \in A} \mathbf{I}^\alpha), \{\lambda^\alpha : \alpha \in A\})$ *and* $(\mathbb{A}^2(\bigcup_{\alpha \in A} \mathbb{J}(\mathbf{I}^\alpha)), \{\lambda^\alpha : \alpha \in A\})$ *are coproducts in the category of complete, completely distrubitive lattices, and complete lattice homomorphisms.*

2.4 Coproducts in MA

Let $\mathbf{L}$ be a complete modular algebraic lattice and let $\mathbf{I}^1, \ldots, \mathbf{I}^N$ be an indexed family of complete totally ordered lattices with completely dense join irreducibles.

Theorem 10. *A bounded lattice homomorphism* $f : \mathbb{S}(\prod_{\alpha \in \{1,\ldots,N\}} \mathbf{I}^\alpha_{>0}) \to \mathbf{L}$ *extends to a complete lattice homomorphism* $g : \mathbb{A}(\prod_{\alpha \in \{1,\ldots,N\}} \mathbb{J}(\mathbf{I}^\alpha)) \to \mathbf{L}$ *if and only if* f *preserves existing meets and joins.*

Proof. In one direction, if f extends to a complete lattice homomorphism then it preserves existing meets and joins. The converse follows from Theorem 33. $\square$

Theorem 11. *Let $f^1 : \mathbf{I}^1 \to \mathbf{L}$ and $f^2 : \mathbf{I}^2 \to \mathbf{L}$ be complete lattice homomorphisms, and let $f : \mathbb{S}(\mathbf{I}^1_{>0} \times \mathbf{I}^2_{>0}) \to \mathbf{L}$ be the bounded modular copairing of f^1 and f^2 relative to the canonical coprojections on $\mathbb{S}(\mathbf{I}^1_{>0} \times \mathbf{I}^2_{>0})$. The following are equivalent.*

1. *One has*
$$\bigwedge_{b \in B} (f_a \vee f_b) = f_a \vee \bigwedge_{b \in B} f_b$$
for each $a \in \mathbf{I}^1 \cup \mathbf{I}^2$ and each subset B of either $\mathbf{I}^1$ or $\mathbf{I}^2$.
2. *Map f preserves existing meets and joins.*
3. *Map f extends to a complete lattice homomorphism $\mathbb{A}(\mathbb{J}(\mathbf{I}^1) \times \mathbb{J}(\mathbf{I}^2)) \to \mathbf{L}$.*

Proof. Equivalence of 2 and 3 is the content of Theorem 10. Condition 2 clearly implies 1, and the converse follows from Lemma 29. $\square$

Theorem 12. *If $\mathbf{I}^1, \mathbf{I}^2$ are complete totally ordered sublattices of a modular algebraic lattice $\mathbf{L}$, then $\mathbf{I}^1 \cup \mathbf{I}^2$ extends to a complete, completely distributive sublattice iff*

$$\bigwedge_{b \in B} (a \vee b) = a \vee \bigwedge_{b \in B} b$$

for each element $a \in \mathbf{I}^1 \cup \mathbf{I}^2$ and each set B contained in $\mathbf{I}^1$ or in $\mathbf{I}^2$.

Proof. Every complete chain in an algebraic lattice has completely dense irreducibles. Apply Theorem 11 to the inclusions $f^1 : \mathbf{I}^1 \to \mathbf{L}$ and $f^2 : \mathbf{I}^2 \to \mathbf{L}$. $\square$

Remark 2. *Theorem 12 fails when the condition that $\mathbf{I}^1$ and $\mathbf{I}^2$ be complete sublattices is dropped. It is simple to construct a counterexample, for instance, when every element of $\mathbf{I}^1$ has finite height but $\mathbf{I}^1$ is countably infinite.*

Theorem 13. *The following are equivalent for any pair of totally ordered sets $\mathbf{I}^1, \mathbf{I}^2$.*

1. *Sets $\mathbf{I}^1$ and $\mathbf{I}^2$ are well ordered.*
2. *Sets $\mathbf{I}^1$ and $\mathbf{I}^2$ have a completely distributive coproduct in* MA.
3. *The map $\mathbb{S}(\mathbf{I}^1 \times \mathbf{I}^2) \to \mathbb{A}(\mathbb{J}(\mathbf{I}^1) \times \mathbb{J}(\mathbf{I}^2))$ defined $S \mapsto S \cap \mathbb{J}(\mathbf{I}^1) \times \mathbb{J}(\mathbf{I}^2)$ is an isomorphism, and the cospan of canonical filtrations $\mathbf{I}^1 \longrightarrow \mathbb{A}(\mathbb{J}(\mathbf{I}^1) \times \mathbb{J}(\mathbf{I}^2)) \longleftarrow \mathbf{I}^2$ is a coproduct in* MA.

Proof. Follows directly from Theorems 11 and 28. $\square$

Remark 3. *The results of this section extend strictly beyond what can be deduced from the structure of free lattices. For example, $\mathbf{J} = (\boldsymbol{R} \cup \{\infty, -\infty\}) \times \{-, +\}$ is a complete totally ordered lattice, and is not free with respect to the forgetful functor into partially ordered sets. This can be verified via Tunnicliffe (1985) Theorem 3, noting that $\mathbb{J}(\mathbf{J}) \cap \mathbb{J}^*(\mathbf{J}) = \emptyset$ fails to generate $\mathbf{J}$. Nevertheless, $\mathbf{J}$ has completely dense irreducibles, and Theorem 10 applies.*

2.5 Coproducts in UD

Theorem 14. *Let $\mathbf{I}^1, \ldots, \mathbf{I}^N$ be well ordered sets with maxima. Let $Q = \prod_m \mathbb{J}(\mathbf{I}^m)$, and for each m let $\lambda^m : \mathbf{I}^m \to \mathbb{A}(Q)$ be the canonical filtration of $\mathbb{A}(Q)$ by $\mathbb{A}(\mathbb{J}(\mathbf{I}^m))$. Then the pair $(\mathbb{A}(Q), \{\lambda^m : m \leq N\})$ is a coproduct of the indexed family $\mathbf{I}$ in*

1. *the category of complete distributive upper continuous lattices and (co)limit preserving lattice homomorphisms.*
2. *the category of bounded distributive lattices and bounded lattice homomorphisms*

Proof. Claim 2 holds *a fortiori* by Theorem 5, Lemma **??**, and Theorem 28. For claim 1, let $\mathbf{L}$ be any complete distributive upper continuous lattice. It suffices to show that the unique map $f : \mathbb{A}(Q) \to \mathbf{L}$ induced by a family of complete lattice homomorphisms $f^\alpha : \mathbf{I}^\alpha \to \mathbf{L}$ is also complete. Preservation of meets holds trivially by Theorems 28 and 29, noting that every element of $\mathbf{I}^m$ is meet irreducible. Preservation of joins holds by Theorem 29 and upper continuity. □

3 Technical results

The content of this section relates closely to well known results on distributive lattices and representations, for example, vis-a-vis canonical extensions and Stone-Priestly duality [2, 4]. We would be a little surprised if any portions were substantially novel, but as we have been unable to find a reference, we supply independent arguments. Much of this work can be organized, conceptually, around the following unifying goal: given a lattice $\mathbf{K}$ with completely dense irreducibles, construct an explicit embedding from $\mathbf{K}$ to a doubly algebraic lattice $\mathbf{L}$. This embedding is not a canonical extension, in general, for it is an easy exercise, in what follows, to construct an example where the finiteness condition of [2, Theorems 2.3 and 2.6] is violated, even when the associated lattices are totally ordered.

3.1 Conventions

An element j is *completely join irreducible* if $\bigvee S = j$ implies $j \in S$. In this case the strict lower bounds of j contain a unique greatest element, denoted $\mathrm{pred}(j) := \bigvee \mathring{\downarrow}(j)$.

A lattice $\mathbf{L}$ has *completely dense join irreducibles* if each element of $\mathbf{L}$ can be expressed as the join of a (possibly empty, possibly infinite) set of completely join irreducibles. It has *completely dense meet irreducibles* if the dual lattice $\mathbf{L}^*$ has completely dense join irreducibles, and *completely dense irreducibles* if it has completely dense join and meet irreducibles. A complete *filtration* $F : \mathbf{I} \to \mathbf{L}$ has completely dense irreducibles if the complete chain $F(\mathbf{I}) = \{F_i : i \in \mathbf{I}\}$ has dense irreducibles.

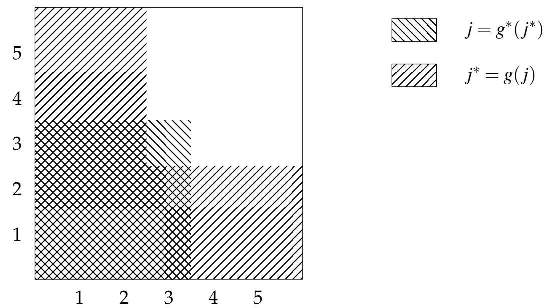

Figure 1: The canonical matching of a completely join irreducible element $j =\downarrow (3,3)$ with a completely meet irreducible element $j^* = \lfloor(2,2)\rfloor = \mathbf{I}^2 - \uparrow(3,3)$ in a distributive lattice $\mathbf{L} = \mathbb{A}(\{1,\ldots,5\}^2)$ with completely dense irreducibles. Since $\mathrm{pred}(j) = j \wedge j^*$ and $\mathrm{succ}(j^*) = j \vee j^*$, one has $g^*(j^*) = j$ and $g(j) = j^*$.

3.2 Irreducible matchings and lattice extensions

Example 1 and its associated Figure 1 may aid intuition in parsing Proposition 15. We refer to the maps g and g^* defined in this context as the *irreducible matchings* of $\mathbf{L}$.

Proposition 15. *Let* $\mathbf{L}$ *be a bounded lattice with completely dense irreducbiles. For each* $j \in \mathbb{J}(\mathbf{L})$ *and each* $j^* \in \mathbb{J}^*(\mathbf{L})$ *there exists exactly one* $g(j) \in \mathbb{J}^*(\mathbf{L})$ *and one* $g^*(j^*)$ *such that*

$$g(j) \wedge j = \mathrm{pred}(j) \qquad\qquad g^*(j^*) \vee j^* = \mathrm{succ}(j^*).$$

The maps g *and* g^* *determine mutually inverse bijections between* $\mathbb{J}(\mathbf{L})$ *and* $\mathbb{J}^*(\mathbf{L})$.

Proof. Let $j \in \mathbb{J}(\mathbf{L})$ be given. Density of meet irreducibles implies that the set difference $S = \mathbb{J}^*(\mathbf{L})_{\geq \mathrm{pred}(j)} - \mathbb{J}^*(\mathbf{L})_{\geq j}$ is nonempty. Fix $j^* \in S$ and note that $\mathrm{pred}(j) \leq j \wedge j^* \leq \mathrm{pred}(j)$, hence $j \wedge j^* = \mathrm{pred}(j)$. For any $s \in S$ one has $s \geq \mathrm{pred}(j) = \bigwedge(\mathbb{J}^*(\mathbf{L})_{\geq j} \cup \{j^*\})$, so irreducibility implies $s = j^*$. Therefore g is well and uniquely defined, as is g^*, by duality. Since $\mathrm{succ}(g(j))$ is the unique element of $\mathbf{L}$ at height 1 above $g(j)$, the following diamond isomorphism confirms that g^* and g are mutually inverse.

$$\begin{tikzcd}
 & \mathrm{succ}(g(j)) \arrow[dl, dash] \arrow[dr, dash] & \\
j \arrow[dr, dash] & & g(j) \arrow[dl, dash] \\
 & \mathrm{pred}(j) &
\end{tikzcd}$$

$\square$

Lemma 16. *Let S be any join-dense subset of a lattice $\mathbf{L}$, and let $a = (a^\beta)_{\beta \in B}$ be any indexed family of elements in $\mathbf{L}$. The following are equivalent.*

1. *Meet $\bigwedge a$ exists.*
2. *Join $\bigvee(\bigcap_{\beta \in B} S_{\leq a^\beta})$ exists.*
3. *Both exist, and $\bigwedge a = \bigvee(\bigcap_{\beta \in B} S_{\leq a^\beta})$.*

Remark 4. *It is worth noting that the following applies to* all *lattices with completely dense irreducibles, including lattices that are not complete. Maps ζ and ζ^* extend naturally to set functions between power set lattices $\mathbb{P}(\mathbb{J}(\mathbf{L}))$ and $\mathbb{P}(\mathbb{J}^*(\mathbf{L})^*)$. Under these extensions, it is simple to check that $\zeta\zeta^*$ is the closure operator on $\mathbb{P}(\mathbb{J}(\mathbf{L}))$ sending S to $\downarrow S$. A dual characterization holds for $\zeta^*\zeta$.*

Remark 5. *The following maps have an interesting parallel in the notion of cones and dual cones.*

Proposition 17. *Let $\mathbf{L}$ be a bounded distributive lattice with completely dense irreducibles, and let g and g^* denote the associated matchings.*

1. *Lattice $\mathbf{L}$ has a compact base, and an element is compact iff it is a join of finitely many completely join irreducibles.*
2. *Lattice $\mathbf{L}$ has a cocompact cobase, and an element is cocompact iff it is a meet of finitely many completely meet irreducibles.*
3. *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{L} \arrow[r, leftrightarrow, "="] \arrow[d, "\xi"'] & \mathbf{L}^* \arrow[d, "\xi^*"] \\
\mathbb{A}\mathbb{J}(\mathbf{L}) \arrow[r, shift left, "\zeta"] & \mathbb{A}\mathbb{J}(\mathbf{L}^*) \arrow[l, shift left, "\zeta^*"]
\end{tikzcd}$$

where

(a) *Each arrow is injective.*
(b) *Vertical arrows preserve existing meets and joins, and horizontal arrows reverse existing meets and joins.*
(c) *By definition, $\xi(a) = \mathbb{J}(\mathbf{L})_{\leq a}$ and $\xi^*(a) = \mathbb{J}(\mathbf{L}^*)_{\leq a}$. Also by definition,*

$$\zeta(S) = g(\mathbb{J}(\mathbf{L}) - S) = \{j^* \in \mathbb{J}^*(\mathbf{L}) : j \leq j^* \text{ for all } j \in S\}$$
$$\zeta^*(S) = g^*(\mathbb{J}^*(\mathbf{L}) - S) = \{j \in \mathbb{J}(\mathbf{L}) : j \leq j^* \text{ for all } j^* \in S^*\}.$$

Proof. Fix $j^* \in \mathbb{J}^*(\mathbf{L})$ and set $j = g(j^*)$. We claim that $j = \min \mathbb{J}(\mathbf{L})_{\not\leq j^*}$. To see this, fix $j' \in \mathbb{J}(\mathbf{L})$ such that $j^* \vee j' > j^*$ and $j' \not\geq j$. Then by distributivity $j^* \vee (j \wedge j') = (j^* \vee j) \wedge (j^* \wedge j') \geq \mathrm{succ}(j^*)$. Since $j \wedge j^*$ is a join of completely join irreducibles, there exists a completely join irreducible j'' such that $j'' < j$ and $j^* \vee j'' = \mathrm{succ}(j^*)$. But only one completely join irreducbile that satisfies the latter identity, by Proposition 15, and this is $g(j^*) = j$. Thus we have a contradiction, and the claim follows.

Consequently, one has $\mathbb{J}(\mathbf{L})_{\leq j^*} = \mathbb{J}(\mathbf{L})_{\not\geq j}$. Since every down-closed subset $S \subseteq \mathbb{J}(\mathbf{L})$ ca be expressed as the set complement of a family of sets of form

$\mathbb{J}(\mathbf{L})_{\not\leq j}$, eg, by letting j run over all elements not contained in S, it follows that the set complement of every down-closed subset $S \subseteq \mathbb{J}(\mathbf{L})$ is an up-closed subset of form $g^*(T) \subseteq \mathbb{J}(\mathbf{L})$, in particular $T = g(\mathbb{J}^*(\mathbf{L}) - S)$. Thus ζ and ζ^* are mutually inverse antiisomorphisms, and the diagram commutes.

Maps ξ and ξ^* preserves existing meets by Lemma 16. Thus they preserve existing joins, by commutativity. All maps are clearly injective, so this completes the proof of all claims relating to the diagram.

For claim 1, note that 4 implies the join irreducible elements of $\mathbf{L}$ are compact. So, too are finite joins of these elements. An element that cannot be expressed as a finite join of completely join irreducibles can nevertheless be expressed as an infinite one, and therefore fails to be compact. Claim 1 follows. Claim 2 is dual. $\square$

Example 1. *Suppose that* $\mathbf{I} = \{1, \ldots, N\}$ *for some positive integer* N*. Then* $\mathbf{I}^2$ *has completely dense irreducibles,*

$$\mathbb{J}(\mathbf{I}^2) = \{\downarrow (m,n) : m,n \in \mathbf{I}\} \qquad \mathbb{J}^*(\mathbf{I}^2) = \{\mathbf{I}^2 - \uparrow (m,n) : m,n \in \mathbf{I}\}$$

and

$$g(\downarrow (m \times n)) = \mathbf{I}^2 - \uparrow (m,n) \qquad g^*(\mathbf{I}^2 - \downarrow (m \times n)) = \downarrow (m \times n).$$

See Figure 1 for illustration.

3.3 Semitopologies on chain products

Let $\mathrm{Cov}(P)$ denote the covering relation on P, that is, the family of ordered pairs (p,q) such that q covers p in P. A *cut* of P is an ordered pair $c = (c_\bullet, c^\bullet)$ such that (i) $c_\bullet$ is down-closed, (ii) $c^\bullet$ is up-closed, (iii) $c_\bullet \cap c^\bullet = \emptyset$, and (iv) $c_\bullet \cup c^\bullet = P$.

Lemma 18. *For any totally ordered set* $\mathbf{I}$*, one has*

$$\mathbb{J}(\mathbf{I}) = \{b : (a,b) \in \mathrm{Cov}(\mathbf{I})\} \qquad \mathbb{J}^*(\mathbf{I}) = \{a : (a,b) \in \mathrm{Cov}(\mathbf{I})\}.$$

Proof. An element is completely join irreducible iff it covers exactly one element, and completely meet irreducible iff it is covered by exactly one element. $\square$

Corollary 19. *The following are equivalent for any totally ordered set* $\mathbf{I}$*.*

1. *Lattice* $\mathbf{I}$ *has a compact base.*
2. *Lattice* $\mathbf{I}$ *has completely dense* join *irreducibles.*
3. *Lattice* $\mathbf{I}$ *has completely dense* meet *irreducibles.*
4. *Each* $i \in \mathbf{I}$ *may be expressed in form*

$$\bigvee(b : (a,b) \in c_\bullet) = \bigwedge(a : (a,b) \in c^\bullet)$$

   for some cut c *of* $\mathrm{Cov}(\mathbf{I})$*.*
5. *For any* $a,b \in \mathbf{I}$*, one has* $a < b$ *iff* $a \leq a' \lessdot b' \leq b$ *for some* $a', b' \in \mathbf{I}$*.*

Proof. All directions follow from Lemma 18. $\square$

Let $\mathbf{I} = (\mathbf{I}^\alpha)_{\alpha \in A}$ be an indexed family of bounded totally ordered sets. Let $\mathbf{J}^\alpha$ be the family of completely join irreducible elements in $\mathbf{I}^\alpha$, and assume that every element of $\mathbf{I}^\alpha$ can be expressed as a join of elements in $\mathbf{J}^\alpha$. Likewise, let $\mathbf{M}^\alpha$ be the family of completely meed irreducible elements of $\mathbf{I}^\alpha$. By Corollary 19, set $\mathbf{M}^\alpha$ is meet dense in $\mathbf{I}^\alpha$. Put

$$\mathbf{H} = \textstyle\prod_\alpha \mathbf{I}^\alpha_{>0} \qquad\qquad Q = \prod \mathbf{J}$$

Lemma 20 (Finite expressibility)**.** *Suppose that index set A is finite. Then to every $X \in \mathbb{S}(\mathbf{H})$ correspond finite subsets $S, T \subseteq \prod_\alpha \mathbb{S}(\mathbf{I}^\alpha)$ such that such that $X = \lceil S \rceil = \lfloor T \rfloor$.*

Proof. Existence of S holds by definition. With regard to T, fix a suitable S and observe that, also by definition, $X = \lceil S \rceil = \bigvee_{s \in S} \bigwedge_{\alpha \in A} \lambda^\alpha(s^\alpha)$. By complete distributivity of the power set lattice, therefore, $X = \bigwedge \{ \bigvee_{s \in S} \lambda^{\alpha_s}(s^{\alpha_s}) : \alpha_s \in A \ \forall\, s \in S\}$. The desired conclusion follows, since each term $\bigvee_{s \in S} \lambda^{\alpha_s}(s^{\alpha_s})$ can be realized in form $\lfloor t \rfloor$ for some $t \in \prod_\alpha \mathbb{S}(\mathbf{I}^\alpha)$. $\square$

Proposition 21. *Let $S \in \mathbb{S}(\mathbf{H})$ be given. The following are equivalent.*

1. *Element S is completely join irreducible in $\mathbb{S}(\mathbf{H})$.*
2. *Index set A is finite, and $S = \downarrow(j)$ for some $j \in \prod \mathbf{J}$.*

Proof. Suppose S is completely join irreducible. If S contains no maximum element, then every element of $T = \{\downarrow(i) : i \in S\}$ is a strict lower bound of S, and $\bigvee T = S$, a contradiction. Therefore S contains a unique maximum, j, and it is elementary to argue that j^α is completely join irreducible for each $\alpha \in A$. For each $\alpha \in A$ define $k(\alpha) \in \prod \mathbf{I}$ by

$$k(\alpha)^\beta = \begin{cases} j^\beta & \alpha \neq \beta \\ \mathrm{pred}(j^\beta) & \alpha = \beta. \end{cases}$$

Then $S - \{j\} = \downarrow_{\mathbf{H}} K$, where K is the antichain $\{k(\alpha) : \alpha \in S\}$. If A is infinite then K is infinite, and it is simple to argue that $\downarrow_{\mathbf{H}} K \notin \mathbb{S}(\mathbf{H})$, whence $\bigvee_{\alpha \in A} \downarrow_{\mathbf{H}} k(\alpha) = S$ a contradiction. Therefore A is finite. This establishes one direction. Conversely, suppose 2. One can define K as before, and it is simple to show $\mathrm{pred}(S) = \downarrow_{\mathbf{H}} K$. $\square$

Corollary 22. *If index set A is infinite then $\mathbb{J}(\mathbb{S}(\mathbf{H}))$ is empty.*

Lemma 23. *Suppose that index set A is finite, and fix $i \in \mathbf{H}$ and $S \in \mathbb{S}(\mathbf{H})$. Then $i \in S$ if and only if $(\prod \mathbf{J})_{\leq i} \subseteq S$.*

Proof. One direction is clear. For the converse, suppose that $(\prod \mathbf{J})_{\leq i} \subseteq S$. We must show that $i \in S$. Let $T = \max(S)$, so that $S = \downarrow(T)$, and assume, without loss of generality, that $A = \{1, \ldots, N\}$. We claim that for each $M \leq N$ there exists a $k \in S$ such that $k^m = i^m$ for all $1 \leq m \leq M$. The proof proceeds by induction on M. The base case $M = 0$ holds because $i^\alpha > 0$ for all α, and each $\mathbf{I}^\alpha$ has completely dense join irreducibles, whence $(\prod \mathbf{J})_{\leq i}$ and therefore S is nonempty. The inductive step follows from density of irreducibles and the fact that T is finite. $\square$

Proposition 24. *Suppose that index set A is finite.*

1. *Lattice $\mathbb{S}(\mathbf{H})$ has completely dense irreducibles, and*

$$\mathbb{J}(\mathbb{S}(\mathbf{H})) = \{\lceil j \rceil : j \in \prod \mathbf{J}\} \qquad \mathbb{J}^*(\mathbb{S}(\mathbf{H})) = \{\lfloor j^* \rfloor : j^* \in \prod \mathbf{M}\}.$$

2. *The injective lattice homomorphism $\mathbb{S}(\mathbf{H}) \to \mathbb{A}(\prod \mathbf{J})$, $S \mapsto S \cap \prod \mathbf{J}$ preserves existing meets and joins. This map restricts to a poset isomorphism between the completely join irreducible elements of its domain and codomain, respectively.*
3. *Let g and g^* denote the irreducible matchings defined in Lemma 15). Put $\phi(i) = \mathbf{H}_{\leq i}$ and $\phi^*(i) = \mathbf{H}_{\not\geq i}$ for each $i \in \prod \mathbf{I}$. Then the following diagram commutes*

$$\begin{tikzcd}
 & \prod \mathbf{J} \arrow[dl, "\phi"'] \arrow[dr, "\phi^*"] & \\
\mathbb{J}\mathbb{S}(\mathbf{H}) \arrow[rr, shift left, "g"] & & \mathbb{J}^*\mathbb{S}(\mathbf{H}) \arrow[ll, shift left, "g^*"]
\end{tikzcd}$$

Proof. Density and characterization of completely join irreducibles were established in Proposition 21 and Lemma 23. For density and characterization of completely meet irreducibles, fix $j \in \prod \mathbf{J}$ and let $j^* \in \prod \mathbf{M}$ be the unique element such that $j^{*\alpha} \lessdot j$ for all $\alpha \in A$. If

$$S = \mathbf{H} - \uparrow j = \lfloor j^* \rfloor.$$

then S is the maximum element $\mathbb{S}(\mathbf{H})$ such that $\downarrow j \not\leq S$. Since, in addition, $(\prod \mathbf{J})_{\leq \lfloor j^* \rfloor} = \prod \mathbf{J} - \uparrow j$, Lemma 16 implies that every element of $\mathbb{S}(\mathbf{H})$ can be expressed as a meet of elements in $\{\lfloor j^* \rfloor : j^* \in \prod \mathbf{M}\}$. Moreover, since $\lfloor j^* \rfloor$ has a least strict upper bound in $\mathbb{S}(\mathbf{H})$, namely $\lfloor j^* \rfloor \cup \{j\}$, all such elements are completely meet irreducible. This completes the proof of claims 1 and 3. Claim 3 follows immediately from 1 and Proposition 17. □

3.4 Semitopologies on products of well ordered sets

Let us reinstate the notation of §3.3.

For any family of partially ordered sets $(P^\alpha)_{\alpha \in A}$, define $\mathbb{F}(P)$ to be the bounded sublatice of $\mathbb{A}(\prod \mathbf{I})$ generated by the images of all canonical filtrations $\mathbb{A}(P^\alpha) \to \prod P$. Write $\mathbb{S}(\mathbf{H}) \sim \mathbb{A}(Q)$ iff the map $\mathbb{S}(\mathbf{H}) \to \mathbb{A}(Q)$, $S \mapsto S \cap Q$ is an isomorphism.

Lemma 25. *The following are equivalent.*

1. $\mathbb{S}(\mathbf{H}) \sim \mathbb{A}(Q)$
2. $\mathbb{A}(Q) = \mathbb{F}(\mathbf{J})$

Lemma 26. *If $A = \{1, \ldots, N\}$ is finite and $\mathbf{I}^m$ is well ordered for each $m \leq N$, then every element of $\mathbb{S}(\mathbf{H})$ can be expressed in form $\mathbb{S}(\mathbf{H}) - \uparrow U$ for some finite set $U \subseteq Q$.*

Proof. Let $S \in \mathbb{S}(\mathbf{H})$ be given and assume. Then by Lemma 20 there exists a finite antichain $T \subseteq \prod \mathbf{I}$ such that $S = \lfloor T \rfloor$. Without loss of generality, $t^m < 1$ for all $t \in T$ and all $m \leq N$, for otherwise $\lfloor t \rfloor = \mathbb{S}(\mathbf{H})$. Thus $\min(\mathbf{H} - \lfloor t \rfloor) = (\mathrm{succ}(t^m))_{m \leq N} \in Q$ for each $t \in T$. One may take $U = \{\min(\mathbf{H} - \lfloor t \rfloor : t \in T\}$. □

Lemma 27. *Suppose that $A = \{1, \ldots, N\}$ and $\mathbf{I}^m$ is well ordered for each $m \le N$. If $\mathbb{A}(Q) = \mathbb{F}(\mathbf{J})$, then every nonempty chain $C \subseteq \mathbb{A}(Q)$ contains a minimum element.*

Proof. Since $\mathbb{F}(\mathbf{J})$ is complete by hypothesis $T := \bigcap C \in \mathbb{F}(\mathbf{J})$, and Lemma 26 provides a finite set U such that $T = Q - \uparrow U$. If a set S contains T, then the containment is strict iff S contains at least one element of U. Since U is finite, there exists $S \in C$ that contains no element of U. One then has $S = \bigcap C = \min C$. $\square$

Theorem 28. *If $A = \{1, \ldots, N\}$ is finite and $\mathbf{I}^m$ is well ordered for each $m \le N$, then $\mathbb{A}(Q) = \mathbb{F}(\mathbf{J})$.*

Proof. We proceed by induction on N. The base case $N = 1$ is clear. For convenience put $Q_M = \prod_{m \le M} \mathbb{J}(\mathbf{I}^M)$ and let $\mathbf{J}_M = (\mathbf{J}^1, \ldots, \mathbf{J}^M)$.

Let $S \in \mathbb{A}(Q)$ be given. One has a decreasing function

$$f : \mathbf{I}^N \to \mathbb{A}(Q_{N-1}) \qquad\qquad i^N \mapsto \{i \in Q_{N-1} : (i, i^N) \in S\}.$$

By the inductive hypothesis $\mathbb{A}(Q_{N-1}) = \mathbb{F}(\mathbf{J}_{N-1})$. Therefore set $C = \{f_i : i \in \mathbf{I}^N\}$ inherits a well order $\le$ from $\mathbb{A}(Q_{N-1})$, via Lemma 27. It likewise inherits a well order $\preceq$ from $\mathbf{I}^N$. As $\le$ and $\preceq$ are opposites, C must be finite. Since S is a union of products $X \times Y$, where $X \in C$ and Y is a lower set of $\mathbf{I}^N$, it follows that $S \in \mathbb{F}(\mathbf{J})$. Thus $\mathbb{A}(Q) = \mathbb{F}(\mathbf{J})$, which was to be shown. $\square$

3.5 Irreducible criterion for continuity

Let us reinstate the notation of §3.3.

Since $f_{\lceil j \rceil} = \bigwedge_\alpha f(\lambda^\alpha_{j^\alpha})$ and $f_{\lfloor j \rfloor} = \bigvee_\alpha f(\lambda^\alpha_{j^\alpha})$, Theorem 29 can be roughly interpreted to say that f preserves joins iff, in lattice $\mathbf{L}$, certain infinite joins commute with coordinate-wise meets. The corresponding statement for preservation of meets is dual. As a mnemonic device, these statements can be compared to the Schwartz theorem of multivariable calculus, which states that the partial derivatives of any twice-continuously differentiable function $\mathbf{R}^n \to \mathbf{R}$ commute.

Theorem 29. *Suppose that index set A is finite, and let $f : \mathbb{S}(\mathbf{H}) \to \mathbf{L}$ be a bounded lattice homomorphism. Then f preserves existing joins iff*

$$\bigvee_{j \in J(i)} f_{\lceil j \rceil} = f_i$$

for all $i \in \prod \mathbf{I}$, where by definition $J(i) = \prod_\alpha \mathbf{J}^\alpha_{\le i^\alpha}$. Dually, f preserves existing meets iff

$$\bigwedge_{j \in J^*(i)} f_{\lfloor j \rfloor} = f_i$$

for all $i \in \prod \mathbf{I}$, where by definition $J^*(i) = \prod_\alpha \mathbf{M}^\alpha_{\ge i^\alpha}$.

Proof. Combine Lemma 2 with Proposition 24 for joins. Meets are dual. $\square$

Remark 6. *It is often convenient to apply Theorem 29 in a trivially equivalent form: map f preserves existing joins iff $\bigvee_{j \in J(i,\alpha)} f_{\lceil j \rceil} = f_i$ for all $i \in \prod \mathbf{I}$, where by definition $J(i,\alpha)$ is the set of all j such that $j^\alpha \in \mathbf{J}^\alpha_{\leq i^\alpha}$ and $j^\beta = i^\beta$ for all $\beta \neq \alpha$. The corresponding statement for meets is dual.*

The following result is not strictly required for the proof of our main results, but it speaks to the role of dense irreducibles in regimes where a compact base is unavailable. We say that e preserves *existing monotone meets* if $e(\bigwedge C) = \bigwedge e(C)$ for every unbounded chain $C \subseteq \mathbb{S}(\mathbf{I}_{>0} \times \mathbf{J}_{>0})$ such that $\bigvee C$ exists. It preserves existing monotone joins if the induced map $e : \mathbb{S}(\mathbf{I}_{>0} \times \mathbf{J}_{>0})^* \to \mathbf{L}^*$ preserves existing monotone meets.

Proposition 30. *For any lattice homomorphism $f : \mathbb{S}(\mathbf{I}_{>0} \times \mathbf{J}_{>0}) \to \mathbf{L}$, the following are equivalent.*

1. *Map f preserves existing meets and joins.*
2. *Map f preserves existing monotone meets and joins.*

If every complete linear sublattice of $\mathbf{L}$ has dense irreducibles, then these conditions are equivalent to

3. *For every complete chain $C \subseteq \mathbf{H}$ and every $i \in C$,*

$$\bigwedge \{a : (a,b) \in \operatorname{Cov}(f(C)),\ i \leq a\} = f(i) = \bigvee \{b : (a,b) \in \operatorname{Cov}(f(C)),\ b \leq i\}.$$

Proof. Equivalence of 1 and 2 follows from Lemma 29 and Remark 6. For 3, suppose every complete linear sublattice of $\mathbf{L}$ has completely dense irreducibles, and fix a complete chain $C \subseteq \mathbb{S}(\mathbf{I}_{>0} \times \mathbf{J}_{>0})$. If 2 holds then $f(C)$ is a complete chain in $\mathbf{L}$, hence has completely dense irreducibles. Since every linear sublattice of $\mathbb{S}(\mathbf{I}_{>0} \times \mathbf{J}_{>0})$ has dense irreducibles (a consequence of the fact that $\mathbb{S}(\mathbf{I}_{>0} \times \mathbf{J}_{>0})$ embeds into a powerset lattice), condition 3 follows from Lemma 18. A similar argument yields the converse. $\square$

4 Proof of Theorem 10

Let $\mathbf{L}$ be a complete modular algebraic lattice, and $\mathbf{I}^1, \ldots, \mathbf{I}^N$ be complete totally ordered sets with completely dense join irreducibles. Let $f : \prod_\alpha \mathbf{I}^\alpha_{>0} \to \mathbf{L}$ be a lattice homomorphism that preserves existing meets and joins. Extend f to a function $\mathbb{A}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha)) \to \mathbf{L}$ via

$$f(Z) = \bigvee_{S \in \mathbf{S}} f(S) \tag{2}$$

where $\mathbf{S} = \{S \in \mathbb{S}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha)) : S \leq Z\}$.

Remark 7. *It is* important *to recall that $\mathbb{S}(P)$ is the* bounded *sublattice of $\mathbb{A}(P)$ generated by all sets of form $\downarrow p$. Therefore its elements include both the empty set and P.*

Lemma 31. *The following are equivalent.*

1. *Map f is a complete lattice homomorphism.*
2. *For each compact $a \in \mathbf{L}$, the collection of all $Z \in \mathbb{A}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$ such that $a \leq f(Z)$ contains a unique minimum element.*

Proof. Under condition 1, the unique minimum Z such that $a \leq f(Z)$ can be computed as the meet of all Z' with this property, since f preserves arbitrary meets. Conversely, assume 2, and fix $\mathbf{S} \subseteq \mathbb{A}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$. Let a be a compact element of $\mathbf{L}$, and Z be the minimum element of $\mathbb{A}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$ such that $a \leq f(Z)$. Then

$$a \leq \textstyle\bigwedge_{S \in \mathbf{S}} f(S) \iff Z \subseteq S \;\forall\; S \in \mathbf{S} \iff Z \subseteq \bigwedge_{S \in \mathbf{S}} S \iff a \leq f(\bigwedge_{S \in \mathbf{S}} S).$$

Since a is arbitrary, it follows that $f(\bigwedge_{S \in \mathbf{S}} S) = \bigwedge_{S \in \mathbf{S}} f(S)$. Thus f preserves arbitrary meets. It preserves arbitrary joins by fiat. Therefore f is complete, which was to be shown. □

Lemma 32. *Let a be a compact element of $\mathbf{L}$, and fix $Z \subseteq \prod_\alpha \mathbb{J}(\mathbf{I}^\alpha)$. The following are equivalent.*

1. *Set Z is the minimum element of $\mathbb{A}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$ such that $a \leq f(Z)$.*
2. *Set Z is the minimum element of $\mathbb{S}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$ such that $a \leq f(Z)$.*

Proof. Posit condition 1. Then set Z can be expressed in form $\bigcup_{p \in P} \downarrow p$ for some subset $P \subseteq \prod_\alpha \mathbb{J}(\mathbf{I}^\alpha)$. By compactness there exists a finite subset $Q \subseteq P$ such that $a \leq f(\bigcup_{q \in Q} \downarrow q)$. By minimality one must have $Z = \bigcup_{q \in Q} \downarrow q \in \mathbb{S}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$. Condition 1 follows.

Now posit condition 2. Assume, for a contradiction, that there exists $Z' \in \mathbb{A}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$ such that $a \leq f(Z')$ and $Z \not\leq Z'$. Without loss of generality, $Z' < Z$. By an argument similar to that of condition 1, there exists $Z'' \leq Z'$ such that $Z'' \in \mathbb{S}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$ and $a \leq f(Z'')$. This yields the desired contradiction. □

Theorem 33. *Map f extends to a complete lattice homomorphism $\mathbb{A}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha)) \to \mathbf{L}$.*

Proof. We proceed by induction on N. The base case $N = 1$ is clear. Assume the desired conclusion holds for all integers strictly less than N. In particular, for any proper subset of $\{1, \ldots, N\}$, the extended map f restricts to a complete lattice homomorphism $\mathbb{A}(\prod_{\alpha \in A} \mathbb{J}(\mathbf{I}^\alpha)) \to \mathbf{L}$ under the canonical embedding of $\mathbb{A}(\prod_{\alpha \in A} \mathbb{J}(\mathbf{I}^\alpha))$ in $\mathbb{A}(\prod_{\alpha \leq N} \mathbb{J}(\mathbf{I}^\alpha))$. Without risk of ambiguity, write $f(S)$ for the value of f on $S \in \mathbb{A}(\prod_{\alpha \in A} \mathbb{J}(\mathbf{I}^\alpha))$ under this embedding.

For convenience, let

$$\mathbf{X} = \textstyle\prod_{\alpha < N} \mathbb{J}(\mathbf{I}^\alpha) \qquad\qquad \mathbf{Y} = \mathbb{J}(\mathbf{I}^N)$$

Let $Y \in \mathbb{A}(\mathbf{Y})$ be given. The map

$$g : \mathbb{S}(\textstyle\prod_{\alpha < N} \mathbf{I}^\alpha_{>0}) \to [f^N(i), 1] \qquad\qquad X \mapsto f(X) \vee f(Y)$$

preserves existing meets and joins because $\mathbb{S}(\prod_{\alpha<N}\mathbf{I}^\alpha_{>0})$ is completely distributive and f preserves existing meets and joins. Therefore it extends to a complete lattice homomorphism on $\mathbb{A}(\prod_{\alpha<N}\mathbf{I}^\alpha)$, by the inductive hypothesis.

Fix a compact $a \in \mathbf{L}$, and note, vacuously, that $a \vee f(Y) \in [f(Y), 1]$. Thus there exists a minimum element $Z \in \mathbb{A}(\prod_{\alpha<N}\mathbb{J}(\mathbf{I}^\alpha))$ such that $a \vee f(Y) \leq g(Z)$, or equivalently, such that

$$a \leq g(Z) = \bigvee_{X \in B} f(X) \vee f(Y)$$

where B is the family of all subsets of Z that lie in $\mathbb{S}(\prod_{\alpha<N}\mathbb{J}(\mathbf{I}^\alpha)) = \mathbb{S}(\mathbf{X})$. Since a is compact, there exists a finite subset $B' \subseteq B$ such that $a \leq \bigvee_{X\in B'} f(X) \vee f(Y)$. Since $Z = \bigcup B$ is minimal, it follows that $Z = \bigcup B' \in \mathbb{S}(\mathbf{X})$.

Thus there exists a map $\xi : \mathbb{S}(\mathbf{Y}) \to \mathbb{S}(\mathbf{X})$ such that

$$\xi(Y) = \min\{X \in \mathbb{S}(\mathbf{X}) : a \leq f(X) \vee f(Y)\}.$$

In fact we have constructed values for ξ given any $Y \in \mathbb{A}(\mathbf{Y})$, and we will use this fact later, but for the moment the symmetry of notation is convenient. By Lemma X there exists an opposing map $\xi^* : \mathbb{S}(\mathbf{X}) \to \mathbb{S}(\mathbf{Y})$ such that

$$\xi^*(X) = \min\{Y \in \mathbb{S}(\mathbf{Y}) : a \leq f(X) \vee f(Y)\}.$$

It is elementary to check that $\xi^*\xi$ and $\xi\xi^*$ are closure operators, and that ξ and ξ^* determine mutually inverse order reversing bijections between $\mathrm{cl}(\mathbb{S}(\mathbf{Y})) = \{\xi^*(X) : X \in \mathbb{S}(\mathbf{X})\}$ and $\mathrm{cl}(\mathbb{S}(\mathbf{X})) = \{\xi(Y) : Y \in \mathbb{S}(\mathbf{Y})\}$.

Now fix $Y, Y' \in \mathrm{cl}(\mathbb{S}(\mathbf{Y}))$ such that $Y < Y'$. Since ξ is an order-reversing bijection, one has $\xi(Y') < \xi(Y)$ and therefore $\xi(Y') \leq \mathrm{pred}(\xi(Y))$. Thus $Y' \geq \xi^*(\mathrm{pred}(\xi(Y)))$. Since Y' was an arbitrary lower bound of Y, it follows that $\xi^*(\mathrm{pred}(\xi(Y)))$ is the least nontrivial successor to Y in $\mathrm{cl}(\mathbb{S}(\mathbf{Y}))$. Every element of $\mathrm{cl}(\mathbb{S}(\mathbf{Y}))$ with a nontrivial successor has a least nontrivial successor, similarly.

For each integer $m < M := |\mathrm{cl}(\mathbb{S}(\mathbf{Y}))|$, let Y_m denote the unique element of $\mathrm{cl}(\mathbb{S}(\mathbf{Y}))$ at height m over the bottom element $\emptyset \in \mathrm{cl}(\mathbb{S}(\mathbf{Y}))$, and let $X_m = \xi(Y_m)$.

First suppose that M is the countably infinite cardinal, and put $Y_M = \bigvee_{m<M} Y_m$. Since the sequence Y_m is strictly increasing, Y_M is not join irreducible. Therefore $Y_M > \xi^*(\xi(Y_M)) =: Y_* \in \mathrm{cl}(\mathbb{S}(\mathbf{Y}))$. Thus $Y_m < Y_*$ for all but finitely many m. However, for any finite m such that $Y_m < Y_* < Y_M$

$$\xi(Y_M) < \xi(Y_m) < \xi(Y_*) = \xi(Y_M),$$

a contradiction. Therefore M is finite.

Now suppose that M if finite. Setting $Y_M = \mathbf{Y}$, put

$$Z_m = (X_m \times \mathbf{Y}) \cup (\mathbf{X} \times Y_m) \qquad\qquad Z = \bigcap_{m<M} Z_m = \bigcup_{m<M} X_m \times Y_{m+1}.$$

By construction of $X_m = \xi(Y_m)$, one has

$$a \leq \textstyle\bigwedge_{m<M}(f(X_m) \vee f(Y_m)) = \bigwedge_{m<M} f(Z_m) = f(Z).$$

By Lemma 31, it suffices to prove that Z is the minimum element of $\mathbb{A}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$ such that $a \leq f(Z)$. By Lemma 32, it therefore suffices to prove that Z is the minimum element of $\mathbb{S}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$ such that $a \leq f(Z)$.

Therefore suppose, for a contradiction, that $Z' \in \mathbb{S}(\prod_\alpha \mathbb{J}(\mathbf{I}^\alpha))$ satisfies both $a \leq f(Z')$ and $Z' \not\geq Z$. Without loss of generality, $Z' < Z$. Fix $(x,y) \in Z - Z'$, and put $X_* = \{x' : (x',y) \in Z'\}$.

By hypothesis, Z' can be expressed as a finite union $\bigcup_{\alpha=1}^{L} \downarrow (x'_\alpha, y'_\alpha)$. For each index α, either $y_\alpha < y$ or $x_\alpha \in X_*$. Thus, taking $Y_* = \bigvee\{\downarrow y_\alpha : y_\alpha < y\} \in \mathbb{S}(\mathbf{Y})$, one has

$$Z' \subseteq (X_* \times \mathbf{Y}) \cup (\mathbf{X} \times Y_*) := Z''.$$

Fix m such that $Y_m = \xi^*(\xi(Y_*)) \leq Y_*$ and therefore $\xi(Y_m) = \xi(Y_*)$. One then has $a \leq f(Z') \leq f(Z'') = f(X_*) \vee f(Y_*)$. It follows from the definition of ξ that $\xi(Y_*) \subseteq X_*$. Since $x \notin X_*$, therefore,

$$x \notin \xi(Y_*) = \xi(Y_m) = X_m.$$

Since $y \notin Y_*$ one has, *a fortiori*, that $y \notin Y_m$. Therefore $(x,y) \notin Z_m$, hence $(x,y) \notin Z$, a contradiction. The desired conclusion follows. □

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