Decomposition of nonlinear persistence modules

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Abstract

Single parameter persistence is a fundamental notion in applied algebraic topology. However, the problem of adapting a persistence framework to nonlinear coefficients remains largely open. Recently, Patel has proposed a notion of persistence diagram compatible constructible modules valued in abelian and Krull-Schmidt categories. We introduce a corresponding decomposition scheme by reformatting the language of classical persistence into that of projective Puppe exact categories. This construction provides a concrete basis on which to analyze the Patelian persistence diagram via standard machinery of homological algebra.

1 Introduction

Over the course of the past several decades, considerable effort has been devoted to problems of the following form: given a field $\Bbbk$ and a nested sequence of topological spaces $X_0 \subseteq \cdots \subseteq X_N$, what can be deduced of X from the sequence V of vector spaces and linear maps

$$H_*(X_0;\Bbbk) \to \cdots \to H_*(X_N;\,\Bbbk)$$

induced by homology?

In 2002, Edelsbrunner, Letscher, and Zomorodian [8] described an approach that predicates on the notion of a *persistence diagram*. The persistence diagram was later reformulated as a *barcode*, which may be regarded as a set function $b : \mathcal{I} = \{\{n,\ldots,m\} : 0 \le n \le m \le N\} \to \mathbf{Z}_{\ge 0}$ determined by V. This approach has significant computational advantages, as the function b is efficiently vectorizable, simple to store in computer memory, and compatible with learning algorithms. However, the information encoded in a barcode is highly incomplete. To wit, it retains no data regarding the relation of X to another topological space.

A positive step to fill this gap was taken by Carlsson and Zomorodian in 2005 [22], who introduced a novel definition of barcode in relation to a generalization of the following result by Gabriel. Here an *interval diagram* means a sequence of linear maps $U_0 \to \cdots \to U_N$ such that (i) $U_m \in \{0, \Bbbk\}$ for all m, (ii) $U_m = \Bbbk$ whenever there exist $n \le m \le l$ such that $U_n = U_l = \Bbbk$, and (iii) all maps have maximum rank.

Theorem 1 (Gabriel)**.** *Let* $\mathbf{I}$ *be the chain* $\{0,\ldots,N\}$*, endowed with the usual order, and let* E *be the category of finite dimensional vector spaces over a ground field* $\Bbbk$*, and linear maps betweeen them.*

1. *Every* $\mathbf{I}$*-shaped diagram in* E *is isomorphic to a direct sum of interval diagrams.*
2. *If* $(V_m)_{m=0}^k$ *and* $(W_n)_{n=0}^l$ *are indexed families of interval diagrams, then* $\oplus V \cong \oplus W$ *if and only if* $k = l$ *and there exists a permutation on* $\{0, \ldots, k\}$ *such that* $V_m \cong W_{\pi(m)}$ *for all* m*.*

The barcode defined by Carlsson and Zomorodian is the function $b : \mathcal{I} \to \mathbf{Z}_{\geq 0}$ such that $b(I)$ is the number of summands supported on I in any such decomposition of V, and coincides (up to a simple reformulation) with that of Edelsbrunner et al. The contribution of this definition was to relate the persistence diagram to an algebraic structure which could be investigated using the machinery of homological algebra. In particular, by relating b to a decomposition of V, it became possible to query individual bars as they relate to maps $Y \to X_m$ and $X_m \to Z$.

However, the barcode paradigm, too, yields incomplete data as concerns X. The theorem of Gabriel, and its many generalizations, deals almost exclusively with *linear* coefficients. Moreover, as with many results of a Krull-Schmit variety, the decomposition of V into direct summands is not canonical. This complicates both formal interpretation and general efforts at integration with natural category theoretic machinery.

Recently, Patel has introduced several notions of barcode applicable to homology with *nonlinear* coefficients [15]. On a base level, these notions mirror the original definitions of Edelsbrunner, Letscher, Zomorodian, and Robbins, and derives from the application of a Möbius transform to the rank function of a module. As such, it shares several of the same analytical limitations.

The present work proposes a remedy to this problem analogous to that posed by Carlsson and Zomorodian for linear modules. Concretely, we show that the barcode introduced by Patel "counts" the components of V with respect to a certain decomposition scheme, just as the original barcode counts bars in a direct sum decomposition. This scheme is both canonical and natural with respect to some classical constructions in algebraic topology, e.g. the Leray-Serre spectral sequence. Moreover, it admits natural extensions of the original definition of Patel in several directions, removing, for example, the criterion of constructibility.

Our main result appears in Theorem 21. Literature and notation are reviewed in §2 and §3, respectively. Our approach is based on the projective theory of Puppe exact categories, recalled in §6. This theory relies on the structure of free objects in the category of complete, completely distributive lattices and complete lattice homomorphisms, developed in separate work [12] and reviewed in §4. The importance of this machinery in dealing with certain limit arguments is illustrated by example in §5. The principle objects of study, homomorphisms K_f and K^*_f, are introduced in §7. Main results appear in §8.

For readers unaccustomed to certain categories of modular, algebraic, and completely distributive lattices, this treatment may appear slightly terse. A gradual introduction to the main ideas may be found in Appendix A. Some details of continuous and algebraic lattices appear in Appendix B.

2 Literature

The projective p-exact approach to persistence presented in this text was previewed by an analogous matroid theoretic treatment in [10] and [11].

There is ample evidence to suggest that persistence modules valued in a category other than finite-dimensional $\Bbbk$-vector spaces have relevance to data science. The earliest examples of persistent topological structure in data include spaces with torsion, e.g. the Klein bottle [5]. The method of circular coordinates developed in [14], for example, relies explicitly on the use of integer coefficients. Persistence for circle-valued maps has also been considered by Burghelea and Dey in [3] and by Burghelea and Haller in [4].

Work in Floer homology has recently prompted active exploration of the infinite-dimensional case. In [21], Usher and Zhang introduces persistence for Floer homology via a non-Archimedean singular value decomposition of the boundary operator of the chain complex. This work has generated a rapidly growing body of literature [17], including works on autonomous Hamiltonian flows [16] and rational curves of smooth surfaces [2].

The projective theory of Puppe exact categories received an expansive treatment by Grandis in [9]. Puppe exact categories have previously appeared in TDA literature, e.g. with regard to stability [1]. Lattice theoretic structure in persistence modules has also been explored in [6] and [7].

The description of persistence in terms of the fundamental subspaces (kernel and image) has been much discussed. The basics of this analysis appear in the original paper of Robins [19], in the seminal papers by Edelsbrunner and Zomorodian [8], and that of Carlsso and Zomorodian [23] which reformulates the persistence diagram in terms of graded modules.

The *generalized* persistence diagram was introduced for $\mathbf{R}$-parametrized constructible modules valued in Krull-Schmidt categories by Patel in [15]. Patel and McClearly have recently adapted the principles introduced in this text to achieve novel stability results in multiparameter persistence [13].

3 Notation

Partial orders The maximum element of a poset P (if it exists) will be denoted 1_P, or simply 1 if context leaves no room for confusion. The minimum element will be denoted 0_P or 0.

The order relation of a partially ordered set P is denoted $\mathrm{Rel}(P)$. The dual poset, i.e. the partial order on the same ground set, with opposite order, is denoted P^*. The set $\mathrm{Cov}(P)$ is the family of covering relations of P. Formally, this is the collection of all $(p,q) \in \mathrm{Rel}(P)$ such that no $r \in P$ satisfies $p < r < q$. Given any function $f : S \to P$, we write

$$f_{\leq p} = \{s \in S : f(s) \leq p\}.$$

Given partially ordered sets P, Q, the product poset is denoted $P \times Q$. Given $a \in P$, we write

$$\downarrow_P a = \{b \in P : b \leq a\} \qquad \mathring{\downarrow}_P a = (\downarrow_P a) - \{a\}$$
$$\uparrow_P a = \{b \in P : b \geq a\} \qquad \mathring{\uparrow}_P a = (\uparrow_P a) - \{a\}$$

Where context leaves no room for confusion, we drop subscripts from arrows.

An *interval* in P is a subset $Q \subseteq S$ such that $q \in Q$ whenever there exist $p, r \in Q$ such that $p \leq q \leq r$. Given $a, b \in P$, we write $[a,b]$ for the set $\{p \in P : a \leq p \leq b\}$.

Lattices and total orders By a *complete totally ordered lattice* we mean a complete lattice $\mathbf{I}$ that is totally ordered. This is distinct from a *complete totally ordered set*, in that the latter may lack a top or bottom element. A *chain* in a lattice $\mathbf{L}$ is a linear sublattice $C \subseteq \mathbf{L}$. A chain is *bounded* if $\mathbf{L}$ is bounded and $\{0,1\} \subseteq C$. It is *complete* if C is a complete lattice and the inclusion $C \hookrightarrow \mathbf{L}$ preserves arbitrary meets and joins.

A nonzero element a in a lattice $\mathbf{L}$ with 0 is *completely join irreducible* if for each $S \subseteq \mathbf{L}$ such that $a \leq \bigvee S$ there exists $s \in S$ such that $a \leq s$. In this case, if $\mathbf{L}$ is complete, a has a greatest strict predecessor, $\mathrm{pred}(a) = \bigvee \mathring{\downarrow} a$. Dually, a nonidentity element a in a lattice $\mathbf{L}$ with 1 is *completely meet irreducible* if for each $S \subseteq \mathbf{L}$ such that $\bigwedge S \leq a$ there exists $s \in S$ such that $a \leq s$. In this case, if $\mathbf{L}$ is complete, a has a minimum strict successor, $\mathrm{succ}(a) = \bigwedge \mathring{\uparrow} a$. Given any lattice $\mathbf{L}$, $\mathbb{J}(\mathbf{L})$ and $\mathbb{J}^*(\mathbf{L})$ denote the sets of completely join and meet irreducible elements of $\mathbf{L}$, respectively.

A lattice is *upper continuous* or *meet continuous* if for any $a \in \mathbf{L}$ the poset homomorphism $x \mapsto x \wedge a$ preserves upward directed suprema. It is *lower continuous* or *join continuous* $\mathbf{L}^*$ is meet continuous, that is, if for any $a \in \mathbf{L}$ the poset homomorphism $x \mapsto x \vee a$ preserves downward directed infima.

As above, given a partial order P, we write $\mathbb{P}(P)$ for the powerset lattice of P. Likewise, we write $\mathbb{A}(P)$ for the lattice of *Alexandrov closed* subsets of P, that is, the decreasing subsets of P, ordered under inclusion. We write $\mathbb{S}(P)$ for the sublattice of $\mathbb{A}(S)$ whose elements are those members of $\mathbb{A}(P)$ expressible in form $(\downarrow p_0) \cup \cdots \cup (\downarrow p_m)$ for some finite collection $p_0, \ldots, p_m \in P$.

Chain diagrams A *chain diagram in E* is a functor f from a totally ordered set $\mathbf{I}$ to E. The *support* of a chain diagram is $\mathrm{supp}(f) = \{i \in \mathbf{I} : f_i \neq 0\}$.

An *interval diagram* is a chain diagram for which there exists an interval $\mathbf{J} \subseteq \mathbf{I}$ such that $f_i = 0$ for all $i \notin \mathbf{J}$ and $f(i \leq j)$ is iso whenever $i, j \in \mathbf{J}$. In this case $\mathbf{J}$ is the *support type* of f. Every nonzero chain diagram has a unique support type. The zero diagram has every support type. The *object type* of a nonzero interval diagram is the isomorphism class f_i, where i is any element of $\mathrm{supp}(f)$. The *object type* of a zero diagram is the isomorphism class of the zero objects.

4 Free objects

We briefly recall some results from [12].

Theorem 2 ([12]). *If $\mathbf{I}^1$, $\mathbf{I}^2$ are complete totally ordered sublattices of a modular algebraic lattice $\mathbf{L}$, then $\mathbf{I}^1 \cup \mathbf{I}^2$ extends to a complete, completely distributive sublattice iff*

$$\bigwedge_{b \in B} (a \vee b) = a \vee \bigwedge_{b \in B} b$$

for each element $a \in \mathbf{I}^1 \cup \mathbf{I}^2$ and each set B contained in $\mathbf{I}^1$ or in $\mathbf{I}^2$.

Given a poset P, write $\mathbb{B}(P)$ for the family of decreasing subsets $S \subseteq P$ such that both S and $P - S$ are nonempty.

Definition 1. *If P is a partially ordered set and $\mathbb{X} \in \{\mathbb{A}, \mathbb{B}\}$, then the* free embedding $\mu : P \to \mathbb{A}\mathbb{X}(P)$ *is the map defined by $\mu(p) = \{T \in \mathbb{X}(P) : p \notin T\}$ for all $p \in P$.*

Remark 1. *Let $\mu : P \to \mathbb{AX}(P)$ be the free embedding.*

1. *If $\mathbb{X} = \mathbb{A}$ then $\mu(p) = \mathbb{A}(P - \uparrow p)$.*
2. *If $\mathbb{X} = \mathbb{A}$ then μ fails to preserve existing top and bottom elements.*
3. *If $\mathbb{X} = \mathbb{B}$ then μ preserves existing top and bottom elements.*

Theorem 3 (Tunnicliffe 1985, [20]). *Let P be a partially ordered set, $\mathbf{L}$ be a complete, completely distributive lattice, and $f : P \to \mathbf{L}$ be an an order-preserving function. In the following diagram,*

$$\begin{tikzcd} & \mathbb{AX}(P) \arrow[dr, "g"] & \\ P \arrow[ur, "\mu"] \arrow[rr, "f"] & & \mathbf{L} \end{tikzcd}$$

let μ denote the free embedding.

1. *If $\mathbb{X} = \mathbb{A}$ then the diagram commutes for exactly one complete lattice homomorphism g.*
2. *If $\mathbb{X} = \mathbb{B}$ then the diagram commutes for exactly one unbounded-complete lattice homomorphism g.*

In either case, the unique commuting homomorphism satisfies

$$g(S) = \bigvee_{X \in M_S} \bigwedge f(X) = \bigwedge_{Y \in N_S} \bigvee f(Y)$$

where $M_S = \{X \subseteq P : \bigwedge \mu(X) \subseteq S\}$ and $N_S = \{X \subseteq P : S \subseteq \bigvee \mu(X)\}$.

5 Puppe exact categories

Definition 2. *A well-powered category E is* Puppe exact *(or p-exact for short) iff*

1. *E has a zero object, kernels, and cokernels*
2. *every mono is a kernel and every epi is a cokernel*
3. *every morphism has an epi-mono factorization*

Given any arrow $f : A \to B$ in a p-exact category E, we denote the direct image operator $f_\bullet : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$ and the inverse image operator $f^\bullet : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$.

Lemma 4. *Let $f : A \to B$ be any arrow in a p-exact category E.*

1. *Posets $\mathrm{Sub}(A)$ and $\mathrm{Sub}(B)$ are modular lattices.*
2. *The pair $(f_\bullet, f^\bullet)$ is a* modular connection*. Concretely, $f_\bullet$ and $f^\bullet$ are increasing maps, and*

$$f^\bullet f_\bullet(a) = a \vee f^\bullet(0) \qquad\qquad f_\bullet f^\bullet(b) = b \wedge f_\bullet(1)$$

for all $a \in \mathrm{Sub}(A)$ and all $b \in \mathrm{Sub}(B)$. In particular, $(f_\bullet, f^\bullet)$ is a Galois connection.

3. *Consequently, if $S \subseteq \mathrm{Sub}(A)$ and $\bigvee S$ exists, then $\bigvee(f_\bullet S)$ exists, and $f_\bullet(\bigvee S) = \bigvee f_\bullet(S)$. Dually $f^\bullet(\bigwedge S) = \bigwedge f^\bullet(S)$ for any S such that $\bigwedge S$ exists.*

Proof. See [9, p. 48-52]. $\square$

The family of *bounded* modular lattices and modular connections forms a Puppe exact category under the composition rule

$$(g_\bullet, g^\bullet) \circ (f_\bullet, f^\bullet) := (g_\bullet f_\bullet, f^\bullet g^\bullet).$$

Each arrow $f : A \to B$ in this category admits a unique epi-mono factorization of form

$$\begin{tikzcd}
\downarrow f^\bullet 0 \arrow[r, hook, "m"] & A \arrow[r, "f"] \arrow[d, two heads, "q"] & B \arrow[r, two heads, "p"] & \uparrow f_\bullet 1 \\
& \uparrow f^\bullet 0 \arrow[r, "g"] & \downarrow f_\bullet 1 \arrow[u, hook, "n"] &
\end{tikzcd}$$

where

$$\begin{array}{lll}
q_\bullet(x) = x \vee f^\bullet 0 & n_\bullet(y) = y & g_\bullet(x) = f_\bullet(x) \\
q^\bullet(x) = x & n^\bullet(y) = y \wedge f_\bullet 1 & g^\bullet(y) = f^\bullet(y)
\end{array}$$

See [9] for details. We denote this category Mlc. We denote the full subcategory whose objects are *complete* modular lattices by CMlc.

Moreover, Each subobject of a lattice $\mathbf{L}$ in Mlc can be represented by a monomorphism of form $(f_\bullet, f^\bullet) : [0,a]_{\mathbf{L}} \to \mathbf{L}$, where $f_\bullet$ is inclusion and $f^\bullet(b) = a \wedge b$. Every such pair of functions is a monomorphism in Mlc. Thus subobjects are in canonical 1-1 correspondence with intervals $[0,a] \subseteq \mathbf{L}$.

Dually, each quotient object of $\mathbf{L}$ in Mlc can be represented by an epimorphism of form $(f_\bullet, g^\bullet) : \mathbf{L} \to [a,0]$, where $f^\bullet$ is inclusion and $f_\bullet(b) = a \vee b$. Every such pair of functions is an epimorphism in Mlc. Thus quotient objects are in canonical 1-1 correspondence with intervals $[a,1] \subseteq \mathbf{L}$.

Importantly for our story, diagrams make new p-exact categories from old.

Lemma 5. *If E and I are categories and E is p-exact, then the category of diagrams E^I is p-exact.*

Proof. See [9, p. 48]. □

Lemma 6. *Let $\mathbf{C}$ be a p-exact category, and let $\mathbf{C}^I$ be the category of I-shaped diagrams in $\mathbf{C}$.*

1. *A sequence* $0 \longrightarrow \mathrm{K} \xrightarrow{\mathrm{k}} A \xrightarrow{\mathrm{k}^*} \mathrm{K}^* \longrightarrow 0$ *is exact in $\mathbf{C}^I$ if and only if* $0_x \longrightarrow \mathrm{K}_x \xrightarrow{\mathrm{k}_x} A_x \xrightarrow{\mathrm{k}^*_x} \mathrm{K}^*_x \longrightarrow 0_x$ *is exact in $\mathbf{C}$ for all* $x \in \mathrm{Obj}(I)$.
2. *An arrow η in $\mathbf{C}^I$ is mono (respectively, epi) iff η_x is mono (respectively, epi) for all $x \in \mathrm{Obj}(I)$.*
3. *Let $(B^j)_{j \in J}$ be an indexed family of subobjects of a diagram $A \in \mathrm{Obj}(\mathbf{C}^I)$. If $\bigvee_j B^j_x$ exists for all $x \in \mathrm{Obj}(I)$, then $\bigvee_j B^j$ exists and satisfies*

$$\left(\bigvee_j B^j\right)_x = \bigvee_j \left(B^j_x\right)$$

for all $x \in \mathrm{Obj}(I)$. Dually, if $\bigwedge_j B^j_x$ exists for all x, then $\bigwedge_j B^j$ exists and satisfies

$$\left(\bigwedge_j B^j\right)_x = \bigwedge_j \left(B^j_x\right)$$

4. *In particular, the subobject lattice of* $A \in \mathrm{Obj}(\mathbf{C}^I)$ *is complete iff that of* A_x *is complete, for all* $x \in \mathrm{Obj}(I)$.

Proof. That a short exact sequence is exact in $\mathbf{C}^I$ if and only if it is object-wise exact follows in a straight-forward fashion from the universal property of (co)kernels. That an arrow is mono (respectively, epi) iff it is object-wise mono (respectively, epi) follows from this characterization of (co)kernels in $\mathbf{C}^I$, and from the fact that monos and epis in p-exact categories are normal [Grandis p 46]. With this characterization established, it is straight-forward to check the formulae for (arbitrary) join and meet, via the universal properties of subobjects. $\square$

6 Subquotients and kernel duality

Let $\mathbf{I}$ be a totally ordered set, and let

$$f : \mathbf{I} \to \mathrm{CMlc}$$

be a $\mathbf{I}$-shaped diagram of complete modular lattices and modular connections. By Lemma 6, the lattice of subdiagrams of f is complete. For each $t \in \mathbf{I}$, define $\mathrm{K}_f(t)$ to be the maximum subobject of f such that $\mathrm{K}_f(t)_a = 0$ for all $a \geq t$. The resulting set function $\mathrm{K}_f : \mathbf{I} \to \mathrm{Sub}_f$ preserves order, and extends to a complete lattice homomorphism

$$\mathrm{K}_f : \mathbb{A}^2(\mathbf{I}) \to \mathrm{Sub}_f.$$

Recall that K_f denotes both the original map $\mathbf{I} \to \mathrm{Sub}_f$ and its extension to $\mathbb{A}^2(\mathbf{I})$, by convention. Dually, define $\mathrm{K}^*_f(t)$ to be the minimal subobject of f such that $\mathrm{K}^*_f(t)_a = 1$ for all $a \leq t$, and extend this map to a complete homomorphism

$$\mathrm{K}^*_f : \mathbb{A}^2(\mathbf{I}) \to \mathrm{Sub}_f.$$

The relation between K_f and K^*_f merits careful description. Let $\phi : \mathbf{I} \to \mathbf{I}^{\mathrm{op}}$ denote the canonical opposite isomorphism functor, and define an involutive anti-isomorphism functor $\psi : \mathrm{Mlc} \to \mathrm{Mlc}$ by

$$\psi : \mathbf{L} \mapsto \mathbf{L}^{\mathrm{op}} \qquad\qquad (f_\bullet, f^\bullet) \mapsto (f_\bullet, f^\bullet).$$

for each lattice $\mathbf{L}$ and modular connection $(f_\bullet, f^\bullet)$. Then to each functor $f : \mathbf{I} \to \mathrm{Mlc}$ corresponds a unique f^* such that the following diagram commutes

$$\begin{array}{ccc}
\mathbf{I} & \xrightarrow{\quad f \quad} & \mathrm{CMlc} \\
\psi \big\uparrow & & \big\downarrow \phi \\
\mathbf{I}^{\mathrm{op}} & \xrightarrow{\quad f^* \quad} & \mathrm{CMlc}
\end{array}$$

The map $f \mapsto f^*$ determines an anti-isomorphism $\eta : [\mathbf{I}, \mathrm{Mlc}] \cong [\mathbf{I}^{\mathrm{op}}, \mathrm{Mlc}]$. As ϕ is also an anti-isomorphism, there exists a unique map K^*_f such that the rectangle formed by vertical and horizontal arrows in the following diagram commutes, and one may check that this formulation of K^*_f agrees with the former. If, in addition, κ is the isomorphism $\mathbb{A}^2(\mathbf{I}^{\mathrm{op}}) \to \mathbb{A}^2(\mathbf{I})$ defined $\kappa(S) = \{X \in \mathbb{A}(\mathbf{I}) : \mathbf{I} - X \notin S\}$, then by a routine exercise the entire diagram commutes.

$$\begin{tikzcd}
{[\mathbf{I}, \mathrm{Mlc}]} \arrow[rr, "\eta"] & & {[\mathbf{I}^{\mathrm{op}}, \mathrm{Mlc}]} & \\
& \mathbb{A}^2(\mathbf{I}) \arrow[ul, "\mathbb{A}^2(\mathrm{K}^*_f)"'] \arrow[rr, "\kappa"] & & \mathbb{A}^2(\mathbf{I}^{\mathrm{op}}) \arrow[ul, "\mathbb{A}^2(\mathrm{K}_{f^*})"'] \\
\mathbf{I} \arrow[uu, "\mathrm{K}^*_f"] \arrow[ur] & & \mathbf{I}^{\mathrm{op}} \arrow[ll, "\phi"'] \arrow[uu, "\mathrm{K}_{f^*}"'] \arrow[ur] &
\end{tikzcd}$$

Therefore, up to post-composition with an op-isomorphism, K_f and K^*_f represent identical constructions, one applied to f, and the other to f^*. Besides conceptual clarification, this affords several proofs by duality.

Let

$$X = \{\bigvee S : S \subseteq \mu(\mathbf{I})\} \qquad\qquad Y = \{\bigvee S : S \subseteq \mu(\mathbf{I})\}.$$

Lemma 7. *Every element of $\mathbb{A}^2(\mathbf{I})$ has form $\downarrow(U)$ or $\mathring{\downarrow}(U)$ for some $U \in \mathbb{A}(\mathbf{I})$.*

Proof. Fix $S \in \mathbb{A}^2(\mathbf{I})$. It is simple to show that $S = \mathring{\downarrow}(\bigcup S)$ when $\bigcup S \notin S$. $\square$

Lemma 8. *One has $\mathbb{A}^2(\mathbf{I}) = X \cup Y$.*

Proof. Let $T \in \mathbb{A}^2(\mathbf{I})$ be given. If T cannot be expressed in form $\downarrow(U)$ then $\bigcup T \subseteq \mathbf{I}$ contains no maximum, and it is simple to show that $T \in X$. If T can be expressed in form $\downarrow(U)$ for some U, then $T = \bigwedge\{\mu(i) : i \notin U\} \in Y$. $\square$

Lemma 9. *Fix $a^*, a \in \mathbf{I}$ and $z^* \in \mathbb{A}^2(\mathbf{I})$ such that $z^* \leq \mu(a^*) \leq \mu(a)$.*

1. *If $z^* = \mu(t^*)$ then*

$$f(a^* \leq a)^\bullet f(t^* \leq a)_\bullet 1 = f(t^* \leq a^*)_\bullet 1 \vee f(a^* \leq a)^\bullet 0 \tag{1}$$
$$f(a^* \leq a)_\bullet f(a^* \leq a)^\bullet f(t^* \leq a)_\bullet 1 = f(t^* \leq a)_\bullet 1 \tag{2}$$

2. *If $z^* \in X$, then*

$$f(a^* \leq a)_\bullet \mathrm{K}^*_f(z^*)_{a^*} = \mathrm{K}^*_f(z^*)_a \tag{3}$$
$$\mathrm{K}^*_f(z^*)_{a^*} \vee f(a^* \leq a)^\bullet 0 = f(a^* \leq a)^\bullet \mathrm{K}^*_f(z^*)_a \tag{4}$$

3. *If $z^* \in Y$ then*

$$f(a^* \leq a)_\bullet \mathrm{K}^*_f(z^*)_{a^*} \leq \mathrm{K}^*_f(z^*)_a \tag{5}$$
$$\mathrm{K}^*_f(z^*)_{a^*} \vee f(a^* \leq a)^\bullet 0 \leq f(a^* \leq a)^\bullet \mathrm{K}^*_f(z^*)_a \tag{6}$$
$$f(a^* \leq a)^\bullet 0 \vee \bigwedge_{t^* \in T} f(t^* \leq a^*)_\bullet 1 \leq \bigwedge_{t^* \in T} f(a^* \leq a)^\bullet 0 \vee f(t^* \leq a^*)_\bullet 1 \tag{7}$$

   where $T = \{i \in \mathbf{I} : z^* \leq \mu(i) \leq \mu(a^*)\}$. If, in addition, strict equality holds in any one of these estimates, then it holds in all three.

Proof. All conclusions may be deduced from Lemma 11, by duality. To argue independently, Equation (1) follows from

$$f(a^* \leq a)^\bullet f(a^* \leq a)_\bullet f(t^* \leq a^*)_\bullet 1 = f(t^* \leq a^*)_\bullet 1 \vee f(a^* \leq a)^\bullet 0.$$

Equation (2) is self-evident, but may be formally deduced by precomposing both sides of (1) with $f(a^* \leq a)_\bullet$. Equation (3) holds because direct image preserves join. Equation (4) follows from (3) by definition of modular connection.

Estimate (7) is a standard property of lattices. The left and righthand sides are identical to those of (6), by (1). Equation (6) converts to (5) via precomposition with $f(a^* \leq a)_\bullet$, and (5) converts to (6) via precomposition with $f(a^* \leq a)^\bullet$. All conversions preserve strict equality. ☐

Corollary 10. *Equality holds in* (5) - (7) *if lattice $f(a^*)$ is lower-continuous. It may fail when $f(a^*)$ is not lower-continuous.*

Proof. Both claims follow from Corollary 12, by duality. Alternatively, lower-continuity implies strict equality in (7). Example 1 shows that strict equality does not hold, in general. ☐

Example 1. *Equality need not hold in* (5)*,* (6)*, or* (7)*, in general. Let $\{a^*, a\}$ be any set of two formal symbols, and let $\mathbf{I}$ be the total order on $\mathbf{Z} \cup \{a^*, a\}$ such that $p \leq q \leq a^* \leq a$ for any $p, q \in \mathbf{Z}$. Set*

$$f(x) = \begin{cases} \boldsymbol{R}^{\{p \in \mathbf{Z} : p \leq x\}} & x \leq a^* \\ \boldsymbol{R} & x = a \end{cases} \qquad\qquad f(x \leq y)v = \begin{cases} v & x \leq y \leq a^* \\ \sum_{p \leq y} v_p & x \leq y = a. \end{cases}$$

for $\nu = (\nu_p)_{p \leq x} \in f(x)$. If $c = (\emptyset, \mathbf{I})$, then $\mathrm{K}^*_f(c^*_+)_{a^*} = 0$ and $\mathrm{K}^*_f(c^*_+)_a = \boldsymbol{R}$. Thus the righthand side of (5) *vanishes, while the righthand side does not.*

Lemma 11. *Fix $a^*, a \in \mathbf{I}$ and $z \in \mathbb{A}^2(\mathbf{I})$ such that $\mu(a^*) \leq \mu(a) \leq z$.*

1. *If $z = \mu(t)$ for some t, then*

$$f(a^* \leq a)_\bullet 1 \wedge f(a \leq t)^\bullet 0 = f(a^* \leq a)_\bullet f(a^* \leq t)^\bullet 0 \tag{8}$$
$$f(a^* \leq t)^\bullet 0 = f(a^* \leq a)^\bullet f(a^* \leq a)_\bullet f(a^* \leq t)^\bullet 0. \tag{9}$$

2. *If $z \in Y$ then*

$$f(a^* \leq a)^\bullet \mathrm{K}_f(z)_a = \mathrm{K}_f(z)_{a^*} \tag{10}$$
$$f(a^* \leq a)_\bullet \mathrm{K}_f(z)_{a^*} = \mathrm{K}_f(z)_a \wedge f(a^* \leq a)_\bullet 1 \tag{11}$$

3. *If $z \in X$ then*

$$\mathrm{K}_f(z)_{a^*} \leq f(a^* \leq a)^\bullet \mathrm{K}_f(z)_a \tag{12}$$
$$f(a^* \leq a)_\bullet \mathrm{K}_f(z)_{a^*} \leq f(a^* \leq a)_\bullet 1 \wedge \mathrm{K}_f(z)_a. \tag{13}$$
$$\bigvee_{t \in T} f(a^* \leq a)_\bullet 1 \wedge f(a \leq t)^\bullet 0 \leq f(a^* \leq a)_\bullet 1 \wedge \bigvee_{t \in T} f(a \leq t)^\bullet 0. \tag{14}$$

   where $T = \{i \in \mathbf{I} : \mu(a) \leq \mu(i) \leq z\}$. If, in addition, strict equality holds in any one of these estimates, then it holds in all three.

Proof. All conclusions may be deduced from Lemma 9, by duality. To argue independently, Equation (8) holds by functoriality, since the lefthand side is

$$f(a^* \leq a)_\bullet f(a^* \leq a)^\bullet f(a \leq t)^\bullet 0 = f(a \leq t)^\bullet 0 \wedge f(a^* \leq a)_\bullet 1.$$

Equation (9) is self-evident, but may be formally deduced by precomposing both sides of (8) with $f(a^* \leq a)^\bullet$. Equation (10) holds because inverse image preserves meet, and (11) follows by definition of modular connection.

Estimate (14) is a standard property of order lattices. The left and righthand sides are identical to those of (13), by (8). In turn, (13) converts to (12) via precomposition with $f(a^* \leq a)^\bullet$, and (12) converts to (13) via precomposition with $f(a^* \leq a)_\bullet$. All conversions preserve strict equality. $\square$

Corollary 12. *Equality holds in* (12) - (14) *if $f(a)$ is upper-continuous. It may fail if $f(a)$ is not upper-continuous.*

Proof. Both claims follow from Corollary 10, by duality. Alternatively, since direct image preserves join, upper-continuity implies strict equality in (14). Example 2 yields a counterexample to the general case. $\square$

Example 2. *In contrast to* (5)-(7)*, estimates* (12)-(14) *hold with strict equality in every persistence module valued in a category of modules and module-homomorphisms, since the subobject lattices in all such categories are upper-continuous. However, strict equality does not hold in general. For a counter example, one may simply apply the involutive anti-isomorphism η to the chain diagram f defined in Example 1.*

Corollary 13. *If $s^*, s \in \mathbb{A}^2(\mathbf{I})$, and $a^*, a \in \mathbf{I}$ satisfy $s^* \leq \mu(a^*) \leq \mu(a) \leq s$, then*

$$f(a^* \leq a)_\bullet \mathrm{K}^*_f(s^*)_{a^*} = \mathrm{K}^*_f(s^*)_a \quad f(a^* \leq a)^\bullet \mathrm{K}^*_f(s^*)_a = \mathrm{K}^*_f(s^*)_{a^*} \vee \mathrm{K}_f(a)_{a^*}$$
$$f(a^* \leq a)^\bullet \mathrm{K}_f(s)_a \quad = \mathrm{K}_f(s)_{a^*} \quad f(a^* \leq a)_\bullet \mathrm{K}_f(s)_{a^*} = \mathrm{K}^*_f(a^*)_a \wedge \mathrm{K}_f(s)_a.$$

Having established the primitive relations between K_f, K^*_f, and the direct and indirect image operators, we may proceed to compound expressions. For economy of notation, given any $s^*, s \in \mathbb{A}^2(\mathbf{I})$ put

$$\lceil s^*, s \rceil = \mathrm{K}^*_f(s^*) \wedge \mathrm{K}_f(s) \qquad\qquad \lfloor s^*, s \rfloor = \mathrm{K}^*_f(s^*) \vee \mathrm{K}_f(s).$$

Remark 2. *Corollary 14 seems to suggest that some implications can't be inverted. In particular, if everything in sight were iff, then we could deduce that upper-continuous implies lower-continuous.*

Corollary 14. *If $s^* \leq \mu(a^*) \leq \mu(a) \leq s$, then*

$$f(a^* \leq a)^\bullet \lceil s^*, s \rceil_a \;\geq \lceil s^*, s \rceil_{a^*}$$
$$f(a^* \leq a)_\bullet \lfloor s^*, s \rfloor_{a^*} \leq \lfloor s^*, s \rfloor_a.$$

If, in addition, equality holds in (5)-(7) *and* (12)-(14)*, then*

$$f(a^* \leq a)_\bullet \lceil s^*, s \rceil_{a^*} = \lceil s^*, s \rceil_a \tag{15}$$
$$f(a^* \leq a)^\bullet \lfloor s^*, s \rfloor_a \;= \lfloor s^*, s \rfloor_{a^*} \tag{16}$$

Proof. Lemmas 9 and 11 imply that $f(a^* \leq a)_\bullet L_{a^*} \leq L_a$ and $f(a^* \leq a)^\bullet L_a \geq L_{a^*}$ for $L \in \{\mathrm{K}_f(s), \mathrm{K}^*_f(s^*)\}$. The two inequalities follow. To check the equations, assume strict equality in (5)-(7) and (12)-(14). Corollary (13) then provides the second of the following identities, and the modular law provides the third.

$$\begin{aligned} f(a^* \le a)^\bullet \lceil s^*, s\rceil_a &= [f(a^* \le a)^\bullet \mathrm{K}^*_f(s^*)_a] \wedge [f(a^* \le a)^\bullet \mathrm{K}_f(s)_a] \\ &= \Big[\mathrm{K}_f(a)_{a^*} \vee \mathrm{K}^*_f(s^*)_{a^*}\Big] \wedge \mathrm{K}_f(s)_{a^*} \\ &= \mathrm{K}_f(a)_{a^*} \vee \lceil s^*, s\rceil_{a^*}. \end{aligned}$$

Postcomposition with $f(a^* \le a)_\bullet$ yields (15), since $\lceil s^*, s\rceil_a \le f(a^* \le a)_\bullet 1$. Symmetrically, Corollary (13) provides the second of the following identities, and the modular law provides the third.

$$\begin{aligned} f(a^* \le a)_\bullet \lfloor s^*, s\rfloor_{a^*} &= [f(a^* \le a)_\bullet \mathrm{K}^*_f(s^*)_{a^*}] \vee [f(a^* \le a)_\bullet \mathrm{K}_f(s)_{a^*}] \\ &= \mathrm{K}^*_f(s^*)_a \vee \Big[\mathrm{K}_f(s)_a \wedge \mathrm{K}^*_f(a^*)_a\Big] \\ &= \lfloor s^*, s\rfloor_a \wedge \mathrm{K}^*_f(a^*)_a. \end{aligned}$$

Postcomposition with $f(a^* \le a)^\bullet$ yields (16), since $f(a^* \le a)^\bullet 0 \le \lfloor s^*, s\rfloor_{a^*}$. □

Corollary 15. *Suppose that $s^* \le \mu(a^*) \le \mu(a) \le s$, and posit strict equality in* (5)-(7) *and* (12)-(14)*.*

1. *Map $f(a^* \le a)_\bullet$ restricts to a surjection $\downarrow \lceil s^*, s\rceil_{a^*} \longrightarrow \downarrow \lceil s^*, s\rceil_a$.*
2. *Map $f(a^* \le a)^\bullet$ restricts to an injection $\uparrow \lfloor s^*, s\rfloor_a \longrightarrow \uparrow \lfloor s^*, s\rfloor_{a^*}$.*

Proof. The desired conclusion follows from Corollary 14 and from the canonical epi-mono factorization of Mlc. □

Definition 3. *For convenience, given a subobject M of an object A, set*

$$M^{quo} = A/\,M.$$

Remark 3. *The second half of Corollary 15 be interpreted as follows: when equality holds in* (5)-(7) *and* (12)-(14)*, the (unrestricted) inverse image opeartor $g(a \le a^*)^\bullet$ is mono, where $g = \lfloor s^*, s\rfloor^{quo}$.*

Remark 4. *If either a^* or a lies outside the interval (c^*, c) then the criteria of Corollary 14 cannot be satisfied, and its conclusion may fail. For example, if $a^* \le a < c^* \le c$, and $f(t) \ne 0$ iff $t = a$, then $\lceil s^*, s\rceil_{a^*} = 0$ and $\lceil s^*, s\rceil_a \ne 0$, so* (15) *cannot hold.*

Lemma 16. *For all cuts c^* and c,*

$$\lceil c^*_+, c_+\rceil \wedge \lfloor c^*_-, c_-\rfloor = \lceil c^*_-, c_+\rceil \vee \lceil c^*_+, c_-\rceil \tag{17}$$

Dually,

$$\lceil c^*_+, c_+\rceil \vee \lfloor c^*_-, c_-\rfloor = \lfloor c^*_-, c_+\rfloor \wedge \lfloor c^*_+, c_-\rfloor \tag{18}$$

Proof. The sublattice generated by K_f and K^*_f is distributive. □

Corollary 17. *If equality holds in* (5)-(7)*, then*

$$\lceil c^*_+, c_+\rceil \wedge \lfloor c^*_-, c_-\rfloor (a^* \le a)_\bullet$$

is epi. Dually, if equality holds in (12)-(14)*, then*

$$\lceil c^*_+, c_+\rceil \vee \lfloor c^*_-, c_-\rfloor^{quo} (a^* \le a)^\bullet$$

is mono.

Proof. The first claim follows from Lemma 16, equation 17, and Corollary 14, equation 15. The second follows from the respective duals to these identities. $\square$

Remark 5. *In particular,*

$$
\begin{aligned}
f(a^* \leq a)_\bullet \lceil c_+^*, c_+ \rceil \wedge \lfloor c_-^*, c_- \rfloor_{a^*} &= \lceil c_+^*, c_+ \rceil \wedge \lfloor c_-^*, c_- \rfloor_a \\
f(a^* \leq a)^\bullet \lceil c_+^*, c_+ \rceil \vee \lfloor c_-^*, c_- \rfloor_a &= \lceil c_+^*, c_+ \rceil \vee \lfloor c_-^*, c_- \rfloor_{a^*}
\end{aligned}
$$

when the corresponding conditions hold in Corollary 17.

Definition 4. *Given subobjects M, N of an object A in an exact category E, a double slash $M \mathbin{/\!\!/} N$ represents the subquotient $M / (M \wedge N)$. Dually, $M \mathbin{/\!\!/}^* N$ represents $(M \vee N)/N$.*

Lemma 18. *Let i^* and i be completely join irreducible elements of $\mathbf{I}^\pm$, with underlying cuts $c^* = \pi(i^*)$ and $c = \pi(i)$.*

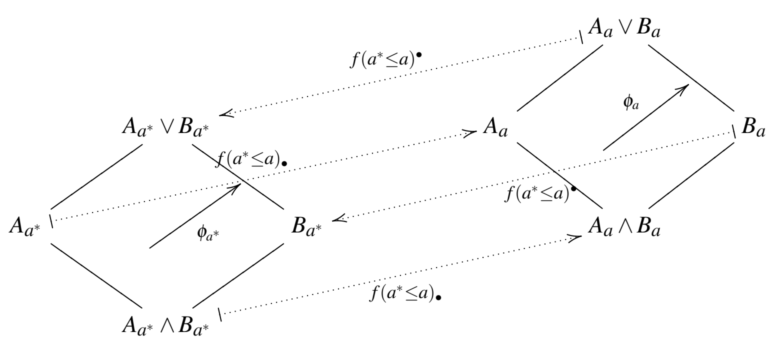
1. *If equality holds in* (5)-(7)*, then the subquotient*

$$\lceil i^*, i \rceil \mathbin{/\!\!/} \lfloor \mathrm{pred}(i^*), \mathrm{pred}(i) \rfloor$$

   is an interval module of support type $(c^*)^\bullet \cap c_\bullet$.
2. *If equality holds in* (12)-(14)*, then the subquotient*

$$\lceil i^*, i \rceil \mathbin{/\!\!/}^* \lfloor \mathrm{pred}(i^*), \mathrm{pred}(i) \rfloor$$

   is an interval module of support type $(c^*)^\bullet \cap c_\bullet$.
3. *Set $A = \lceil i^*, i \rceil$ and $B = \lfloor \mathrm{pred}(i^*), \mathrm{pred}(i) \rfloor$. Let ϕ be the following indexed family of diamond isomorphisms. If equality holds in* (5)-(7)*and* (12)-(14)*, then ϕ is a natural isomorphism $A \mathbin{/\!\!/} B \to A \mathbin{/\!\!/}^* B$. In particular, ϕ is an isomorphism of interval persistence modules.*



Proof. Assume that $i^* = c_+^*$ and $i = c_+$, for otherwise $A = B = 0$, by Lemma **??**. Consequently $\mathrm{pred}(i^*) = c_-^*$ and $\mathrm{pred}(i) = c_-$. Likewise assume $c_+^* \leq c_+$, since otherwise $\mathrm{K}_f^*(c_+^*) \wedge \mathrm{K}_f(c_+) < \mathrm{K}_f^*(c_+^*)$, hence $\mathrm{K}_f^*(c_+^*) \wedge \mathrm{K}_f(c_+) \leq \mathrm{K}_f^*(\mathrm{pred}(c_+^*))$ and again A and B vanish.

For the first proposition, posit (5)-(7)and fix $c^* \leq a^* \leq a \leq c$. Then $f(a^* \leq a)_\bullet A_{a^*} = A_a$ by Corollary 15 and $f(a^* \leq a)_\bullet (A_{a^*} \wedge B_{a^*}) = A_a \wedge B_a$ by Remark 5. The corresponding map of subquotients $[A_{a^*} \wedge B_{a^*}, A_{a^*}] \to [A_a \wedge B_a, A_a]$ has kernel

$$[(A_{a^*} \wedge B_{a^*}) \vee f(a^* \leq a)^\bullet 0] \wedge A_{a^*} = (A_{a^*} \wedge B_{a^*}) \vee [f(a^* \leq a)^\bullet 0 \wedge A_{a^*}].$$

with equality provided by the modular law. Since $f(a^* \leq a)^\bullet 0 \leq \mathrm{K}_f(c_-)_{a^*} \leq B_{a^*}$, the righthand side equals $A_{a^*} \wedge B_{a^*}$. Thus the induced map is epi and mono, hence iso.

For $a > c$ the quotient diagram vanishes, since the numerator $\lceil c_+^*, c_+ \rceil_a$ vanishes. For $a^* < c^*$ one has $\lceil c_+^*, c_+ \rceil_{a^*} \leq f_{a^*} = \lfloor c_-^*, c_- \rfloor_{a^*}$, hence numerator and denominator agree, and again the quotient vanishes.

This establishes the first proposition. The proof of the second is dual. That ϕ is a natural isomorphism follows from coherence (Grandis 2.7.7). □

7 Generalized persistence diagrams

The study of persistent homology serves both as a driver for diverse research initiatives in pure and applied fields of mathematics, and as a substrate where these fields can interact. Until quite recently, the study of persistence focused almost exclusively on homology with field coefficients. However, Patel [15] has recently proposed several notions of persistence diagram suitable for any Krull-Schmidt category and, in particular any abelian one, subject to certain finiteness constraints.

The idea behind these constructions shares the spirit in which the first linear persistence diagrams were cast, eg [8]. A condensed summary appears in §8.1. On a high level, the strategy is to cast the persistence diagram as the Möbius inverse of a so-called *rank function*. The key insight of Patel was a suitable definition of Möbius inverse when the the rank function takes values not in integers, but in objects of a Krull-Schmidt category.

This notion is ground breaking. It offers a vast array of new invariants with which to study filtered complexes, both in pure mathematics and in data science. It likewise lifts the study of persistence modules to a level of abstraction suitable to substantive questions in homological algebra.

A fundamental step toward realizing the potential of this contribution is understanding how to assign semantic meaning to persistence diagrams with nonlinear coefficients. In the case persistent homology with linear coefficients, this question has been answered by overwhelming consensus of the TDA community in terms of the indecomposable factors of the persistence module. We propose that interval factors should assume this rule for the new persistence.

The argument in favor of this motion is arranged as follows. Posit a chain diagram $f : \mathbf{I} \to E$ for some abelian category E, and suppose that a complete lattice homomorphism $\Lambda : \mathbb{A}(\mathbb{J}(\mathbf{I}^{\pm})^2) \to \mathrm{Sub}(f)$ cuts the associated diamond with marginals K_f and K_f^*, coherenetly. We show that the rank function of f

7.1 Background

We review the fundamentals of generalized persistence laid out in [15].

Let Dgm be the poset whose elements are intervals in $\mathbf{R}$ that contain a minimum but no maximum, and whose order is reverse inclusion. Let G be an abelian group with a translation invariant ordering on its elements, and S be a finite set. A map $Y : \mathrm{Dgm} \to \mathrm{G}$ is *S-finite* if $Y(I) \neq 0$ implies $I = [s_i, s_j)$ or $I = [s_i, \infty)$ for some $s_i, s_j \in S$. It is a *Patel persistence diagram* if it is S-finite for some S.

A functor f from $\mathbf{R}$ to an essentially small symmetric monoidal category E with identity object e is *S-constructible* in the sense of [15] if $S = \{s_1 < \cdots < s_n\} \subseteq \mathbf{R}$ is finite and

1. for $p \leq q < s_1$, map $f(p \leq q)$ is identity on e
2. for $s_i \leq p \leq q < s_{i+1}$, map $f(p \leq q)$ is an isomorphism
3. for $s_n \leq p \leq q$, map $f(p \leq q)$ is an isomorphism.

Functor f is *constructible* if it is S constructible for some S.

Let $\mathsf{J}(E)$ be the set of isomorphism classes of E, regarded as a commutative monoid under the the binary monoidal operation $\square$. That is, $[a]+[b] = [a \square b]$. Let $\mathsf{A}(E)$ be the group completion of $\mathsf{J}(E)$. When E is abelian one may define $\mathsf{B}(E)$ as the quotient $\mathsf{A}(E)/\sim$, where $[a] \sim [b]+[c]$ iff there exists a short exact sequence (not necessarily split) $0 \to b \to a \to c \to 0$. Both $\mathsf{A}(E)$ and $\mathsf{B}(E)$ admit translation invariant partial orders compatible with subobject inclusion. See [Patel] for details.

To every S constructible functor f one may associate a map df_B as follows. Select a $\delta > 0$ such that $s_i < s_{i+1} - \delta$ for all $i \in \{1, \ldots, n-1\}$, and any $s' > s_n$. Then

$$df_B(I) = \begin{cases} [\mathrm{Im}(f(p < s_i - \delta))] & I = [p, s_i) \\ [\mathrm{Im}(f(p < s'))] & I = [p, \infty) \\ [\mathrm{Im}(f(p < q))] & \text{all other } I = [p, q) \end{cases}$$

and $df_A(I) = \pi(df_B(I))$, where π is the quotient map $\mathsf{J}(E) \to \mathsf{B}(E)$.

Definition 5 (Patel)**.** *The type A persistence diagram of f is the Möbius inversion $f^A : \mathrm{Dgm} \to \mathsf{A}(E)$ of df_A. If E is Abelian, then the type B persistence diagram of f is the Möbius inversion $f^B : \mathrm{Dgm} \to \mathsf{B}(E)$ of df_B.*

7.2 Results

Let E be an essentially small abelian category where every object has finite height, and let Υ be the set of isomorphism classes of simple objects in E.

As in any category of finite-length ring modules, to each object A in E corresponds a multiset of composition factors $\mathrm{comp}(A)$, regarded as finitely supported set function $\Upsilon \to \mathbf{Z}_{\geq 0}$. One may calculate $\mathrm{comp}(A)$ as the multiset of quotients of form A_i / A_{i-1} for any maximal chain $0 = A_0 \leq \cdots \leq A_m = A$ in the order lattice $\mathrm{Sub}(A)$. This quantity is well defined, see Grandis 6.1.6, p250. Consequently, $\mathsf{B}(E)$ is canonically isomorphic to the family of finitely supported functions $\Upsilon \to \mathbf{Z}$.

Theorem 19. *Let $f : \boldsymbol{R} \to E$ be a constructible persistence module. Then f has an interval code β, and the type-B persistence diagram of f may be expressed*

$$f^B = \phi \beta \psi$$

where $\psi : \mathrm{Dgm} \to \mathrm{Cut}(\mathbf{I})^2$, $[a,b) \mapsto (\varepsilon^a, \varepsilon^b)$ and $\phi(g)$ is the object type of g, regarded as an element of $\mathsf{B}(E)$.

Proof. Since f is constructible, filtrations K_f and K_f^* take only finitely many values. Thus diamond (19) has a complete coherent cut Λ.

$$\begin{tikzcd} & \mathbf{I}^{\pm} \arrow[dl, "\lambda^*"'] \arrow[dr, "\mathrm{K}_f^*"] & \\ \mathbb{P}(\mathbb{J}(\mathbf{I}^{\pm})^2) \arrow[rr, dashed, "\Lambda"] & & \mathrm{Sub}(f) \\ & \mathbf{I}^{\pm} \arrow[ul, "\lambda"] \arrow[ur, "\mathrm{K}_f"'] & \end{tikzcd} \qquad (19)$$

Let $[a,b) \in \mathrm{Dgm}$ be given and, for convenience, write

$$\varepsilon^a = (\mathbf{R}_{<a}, \mathbf{R}_{\geq a}) \qquad x^* = \lambda^*(\varepsilon^a_+) \qquad x = \lambda(\varepsilon^b_+).$$

It is simple to check that $df_A[a,b) = \mathrm{comp}(A)$, where A is the subquotient

$$\mathrm{K}^*_f(\varepsilon^a_+)_a \mathbin{/\!\!/} \mathrm{K}_f(\varepsilon^b_+)_a = (\Lambda x^*)_a /\ (\Lambda(x^* \cap x))_a.$$

Setting $P = x^* - x$, define a complete lattice homomorphism

$$\Gamma_a : \mathbb{A}(P) \to [\Lambda(x^* \cap x)_a, \Lambda(x^*)_a] \cong \mathrm{Sub}(A) \qquad \Gamma_a(T) = \Lambda(T \cup (x^* \cap x))_a.$$

Setting $\gamma(p) = \mathrm{comp}((\Gamma \downarrow p)\ /\ (\Gamma \mathring{\downarrow} p))$, we have $\mathrm{comp}(A) = \sum_{p \in P} \gamma(p)$, by Lemma 20. We claim that γ vanishes on $p = (c^*_{\sigma^*}, c_\sigma)$ when either of the following conditions hold.

1. At least one of σ^* and σ is negative.
2. At least one of $c^*_\bullet$ and $c_\bullet$ has a maximal element.

Indeed, if one of σ^*, σ is negative then $\gamma(p)$ vanishes by Theorem **??**. If $c^*_\bullet$ or $c_\bullet$ has a maximal element then it can be reasoned directly that $\mathrm{K}_f(c_\sigma) = \mathrm{K}_f(\mathrm{pred}(c_\sigma))$ or $\mathrm{K}^*_f(c^*_{\sigma^*}) = \mathrm{K}^*_f(\mathrm{pred}(c^*_{\sigma^*}))$, since f is constructible. In either case $\gamma(p)$ vanishes. An element $p \in P$ violates conditions 1 and 2 iff $p = (\varepsilon^{a'}_+, \varepsilon^{b'}_+)$ for some $a' \leq a$ and some $b' \geq b$. If we denote the set of all such violators by P', then

$$df_A[a,b) = \mathrm{comp}(A) = \sum_{p \in P'} \gamma(p) = \sum_{[a,b) \subseteq [a',b')} \phi(\beta(\varepsilon^{a'}, \varepsilon^{b'})).$$

Thus $\phi\beta\psi$ and df_A are Möbius transforms, which was to be shown. $\square$

Lemma 20. *Let A be an object of finite length in an abelian category E. If P is a partially ordered set and $\Lambda : \mathbb{A}(P) \to \mathrm{Sub}(A)$ is a complete lattice homomorphism, then*

$$\mathrm{comp}(A) = \sum_{p \in P} \mathrm{comp}((\Lambda \downarrow p)\ /\ (\Lambda \mathring{\downarrow} p)).$$

Proof. Let ℓ be any linearization of P, and let $\mathbf{J} = \{\Lambda(S) : S \in \mathbb{A}(\ell)\}$, noting that $\mathbf{J}$ is finite. Fix $(i,j) \in \mathrm{Cov}(\mathbf{J})$ and define $\mathbf{F} = \{S \in \mathbb{A}(\ell) : j \leq \Lambda(S)\}$ and $\mathbf{F}^c = \mathbb{A}(\ell) - \mathbf{F} = \{S \in \mathbb{A}(\ell) : \Lambda(S) \leq i\}$. Then

$$\bigvee \mathbf{F}^c = \max \mathbf{F}^c \qquad \bigwedge \mathbf{F} = \min \mathbf{F} \qquad \bigvee \mathbf{F}^c = \mathrm{pred}_\ell(\bigwedge \mathbf{F}).$$

Since $\bigwedge \mathbf{F}$ has a predecessor, it contains a maximum element $\phi(i,j) \in \ell$. The set function $\phi : \mathrm{Cov}(\mathbf{J}) \to \ell$ is clearly injective. It is simple to check that $(\Lambda \ell_{\leq p}) /\ (\Lambda \ell_{<p}) = 0$ for any p that lies outside the image of ϕ, while for $p = \phi(i,j)$ one has a modular diamond

$$\begin{tikzcd}
 & \ell_{\leq p} \arrow[dl, dash] \arrow[dr, dash] & \\
\ell_{<p} \arrow[dr, dash] & & P_{\leq p} \arrow[dl, dash] \\
 & P_{<p} &
\end{tikzcd} \tag{20}$$

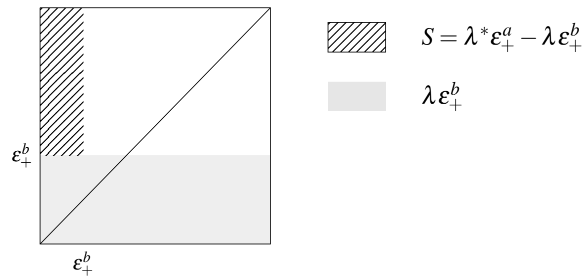


Figure 1: Elements of $\mathbb{A}(\mathbf{I}' \times \mathbf{J}')$, in the special case where $\mathbf{I}' = \{i_1 < i_2 < i_3\} \subseteq \mathbf{I} = \mathbf{R}$ and $\mathbf{J}' = \{i_1 < i_2 < i_3\} \subseteq \mathbf{J} = \mathbf{R}$.

hence subquotient $j/\,i = (\Lambda \ell_{\leq p})/\,(\Lambda \ell_{<p})$ is canonically isomorphic to $(\Lambda P_{\leq p})/\,(\Lambda P_{<p})$. The desired conclusion follows, since $\mathrm{comp}(A) = \sum_{(i,j) \in \mathrm{Cov}(\mathbf{J})} \mathrm{comp}(j/\,i)$. $\square$

Theorem 21. *Let $f : \mathbf{I} \to E$ be a constructible chain diagram in an abelian category E, and let S be the poset $\mathrm{supp}(df_A) = \{[a,b) : df_A[a,b) \neq 0\}$ with the partial order inherited from Dgm. Then each linearization $\ell = \{l^1 < \cdots < l^N\}$ of S corresponds a unique sequence of submodules $0 = g^0 \subseteq \cdots \subseteq g^N = f$ such that*

$$\beta(a^m, b^m) = [g^m /\, g^{m-1}]$$

for all $1 \leq m \leq N$, where $[a^m, b^m) = l^m$.

Proof. By a simple induction, it suffice to show that there exists a unique submodule g^1 such that $\beta(a^1, b^1) = [g^1/g^0] = [g^1]$. Put

$$g^0 = \Lambda \mathring{\downarrow}(\varepsilon_+^{a^1}, \varepsilon_+^{b^1}) \qquad\qquad g^1 = \Lambda \downarrow (\varepsilon_+^{a^1}, \varepsilon_+^{b^1}).$$

Since $[a^1, b^1)$ is minimal in S, the set $S' = \{(\varepsilon_+^a, \varepsilon_+^b) : [a,b) \in S\}$ has trivial intersection with $\mathring{\downarrow}(\varepsilon_+^{a^1}, \varepsilon_+^{b^1})$. Therefore g^0 vanishes, and by definition

$$\beta(a^1, b^1) = [\Lambda \downarrow (\varepsilon_+^{a^1}, \varepsilon_+^{b^1}) /\, \Lambda \mathring{\downarrow}(\varepsilon_+^{a^1}, \varepsilon_+^{b^1})] = [g^1].$$

This establishes existence. For uniqueness, let h be any subdiagram of f such that $[h] = \beta(a^1, b^1)$. Since $\mathrm{supp}(h) = [a^1, b^1)$, one has $h \leq \mathrm{K}_f(\varepsilon_+^{b^1})$ and $h \leq \mathrm{K}_f^*(\varepsilon_+^{a^1})$, hence

$$h \leq \mathrm{K}_f^*(\varepsilon_+^{a^1}) \wedge \mathrm{K}_f(\varepsilon_+^{b^1}) = \Lambda \downarrow (\varepsilon_+^{a^1}, \varepsilon_+^{b^1}) = g^1.$$

If h is a proper submodule then its object type will have composition length strictly less than that of $[g^1] = \beta(a^1, b^1)$, a contradiction. Therefore $h = g^1$. $\square$

Example 3. *Define $h : [0, \infty) \to \mathbf{Z}$ by*

$$h_i = \begin{cases} 0 & 0 \leq i < 1 \\ 4 & 1 \leq i < 2 \\ 2 & 2 \leq i \end{cases}$$

*and let f be the **R** shaped diagram in the category of abelian groups such that $f_i = 0$ for $i < 0$, and*

$$f_i = \mathbf{Z}/\, h_i \qquad\qquad f(i \leq j)_\bullet(x + h_i) = x + h_j$$

for $0 \leq i \leq j$. The values of f on objects are represented schematically in Figure 2.

Filtration K_f^* then takes exactly two values, 0 and f. Filtration K_f takes two nontrivial values, the submodules with support $[0,1)$ and $[0,2)$. Consequently $\mathrm{supp}(df_A) = \{l^1, l^2, l^3\}$, where

$$l^1 = [0,1) \qquad\qquad l^2 = [0,2) \qquad\qquad l^3 = [0,\infty).$$

This set admits a unique linear order $l^1 < l^2 < l^2$ compatible with inclusion. The values of g^p and of the interval modules $g^p \,/\, g^{p-1}$ appear in Figure 2.

The value added by the lattice-theoretic approach is underscored by the fact that we can now associate generators *to the persistence diagram. Suppose, for example, that S is a unit circle, A and B are disks, and $\eta : \partial A \to S$ and $\kappa : \partial B \to S$ are continuous maps with winding numbers 4 and 2, respectively. Let X be the functor from **R** to the category of continuous maps and topological spaces such that*

$$X_i = \begin{cases} \emptyset & i < 0 \\ S & 0 \leq i < 1 \\ S \coprod_\eta A & 1 \leq i < 2 \\ S \coprod_\eta A \coprod_\kappa B & 2 \leq i \end{cases}$$

and $X(i \leq j)$ is inclusion for all $i \leq j$. Postcomposing X with the degree 1 singular homology functor – with integer coefficients – yields a chain diagram isomorphic to f:

$$f \cong H_1 \circ X.$$

Moreover, if ζ is a fundamental class for S, then the nonzero objects in the cyclic quotient modules g^1/g^0, g^2/g^1, and g^3/g^2 are generated by $[4\zeta]$, $[2\zeta]$, and $[\zeta]$, respectively. In the parlance of topological data analysis, these generators represent three "features" of data born at time zero, that vanish at times 1, 2, and ∞, respectively.

A Appendix: order decomposition

Several profound decomposition schemas in pure mathematics rest either wholy or in part on a family of order lattices. In some cases, the associated family constitutes a discipline in its own right:

Measure theory *the decomposition of a ground set into measurable subsets*. By definition, a sigma algebra on S is a complemented sublattice of $\mathbb{P}(S)$ with countable joins.

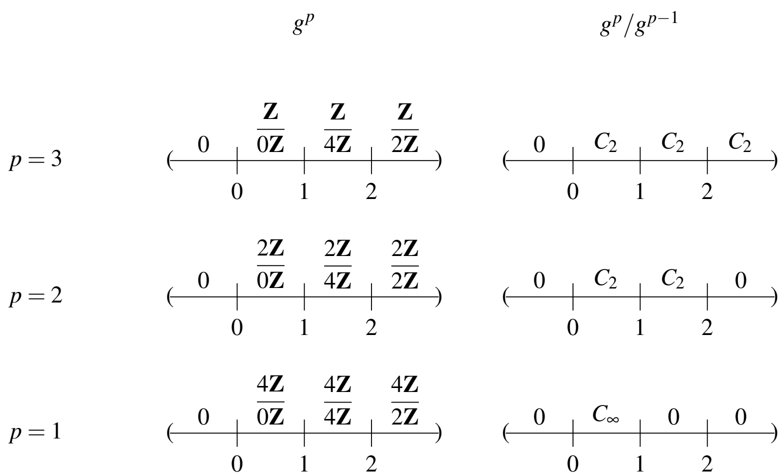


Figure 2: Subquotients of chain diagram $f = g^3$.

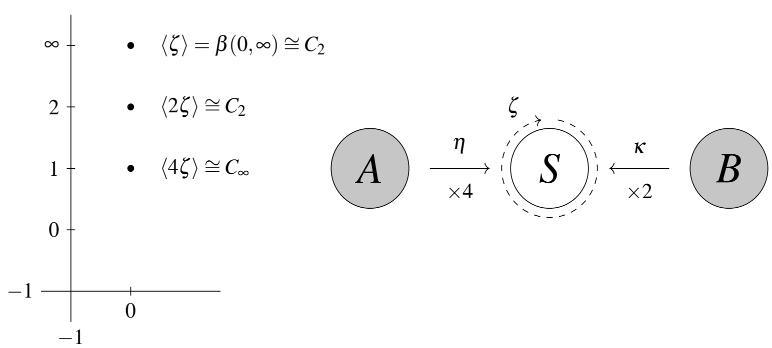


Figure 3: *Left* The Patelian persistence diagram of chain diagram f. *Right* Components of the identification space $S \coprod_\eta A \coprod_\kappa B$. Element $\zeta \in H_1(S\,;\,\mathbf{Z})$ is a fundamental class of S.

Topology *the decomposition of a ground set into open subsets*. By definition, a topology on S is a bounded sublattice of $\mathbb{P}(S)$ with arbitrary joins.

In other cases, multiple lattice families associate to a single branch, each revealing a distinct aspect of its essential character.

Algebra *the decomposition of subobjects.* Let $\mathbb{P}(N)$ be the power set lattice on $\{1,\ldots,N\}$ and $\mathbb{A}(N)$ be the Alexandrov topology on $\{1,\ldots,N\}$, that is, the lattice of down-closed sets. In §X, we will see that for any abelian group G with lattice of subgroups $\mathrm{Sub}(G)$, the internal direct sum decompositions $G \cong \boxplus_i H_i$ correspond naturally to lattice homomorphisms $\mathbb{P}(N) \to \mathrm{Sub}(G)$, while filtrations of G correspond naturally to lattice homomorphisms $\mathbb{A}(N) \to \mathrm{Sub}(G)$.

Thus two elementary notions in algebra – filtration and direct sum – correspond to complete embeddings of two different lattice families, $\mathbb{A}(N)$ and $\mathbb{P}(N)$, into a lattice of subobjects. There are in fact several posets P for which maps $\mathbb{A}(P) \to \mathrm{Sub}(G)$ inform our understanding of G, not only when G is abelian group, but also when it is any object in an exact category. In what follows, we focus primarily on homomorphisms of form

$$\Lambda : \mathbb{X}(P) \to \mathrm{Sub}(V) \tag{21}$$

where V is a module over ring R, $\mathrm{Sub}(V)$ is the lattice of submodules:

$$A \vee B = A + B \qquad\qquad A \wedge B = A \cap B \qquad\qquad (A, B \subseteq V)$$

P is a partially ordered set, and $\mathbb{X}(P)$ is some lattice of subsets of P:

$$S \vee T = S \cup T \qquad\qquad S \wedge T = S \cap T. \qquad\qquad (S, T \subseteq P)$$

We treat two forms of $\mathbb{X}(P)$ in particular: the *power set lattice*

$$\mathbb{P}(P) := \{S : S \subseteq P\}$$

and the *Alexandrov topology*

$$\mathbb{A}(P) := \{S \subseteq P : S = \cup_{s \in S} \downarrow (s)\}$$

It will be useful to recall some examples for particular values of P.

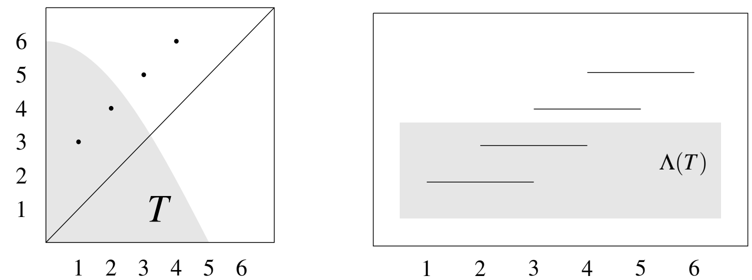
A.1 Ring modules: $P = \{1,\ldots,N\}$

When $P = \{1,\ldots,N\}$ with the canonical order on integers, lattice homomorphisms of type (21) take familiar forms.

A.1.1 Filtration

A (bounded, finite) *filtration* on a module V is a nested sequence of submodules of form $0 = \Lambda_0 \subseteq \cdots \subseteq \Lambda_N = V$. Equivalently, it is a monotone map $\Lambda : \{0,\ldots,N\} \to \mathrm{Sub}(V)$ that preserves top and bottom elements. Since $\mathbb{A}(\{1,\ldots,N\})$ is canonically isomorphic to $\{0,\ldots,N\}$ as an order lattice, it follows that (finite, bounded) filtrations lie in canonical 1-1 correspondence with complete lattice homomorphisms

$$\Lambda : \mathbb{A}(\{1,\ldots,N\}) \to \mathrm{Sub}(V).$$



(a) Persistence diagram of f (b) Barcode of f

Figure 4: By definition, $\Lambda(T) = \Lambda^g(T)$ is the sum of all interval submodules g_s whose birth/death pairs lie in T. The submodules in question for this picture are g_1, supported on $[1,3)$, and g_2, supported on $[2,4)$.

A.1.2 (Internal) direct sum

If $(U_i)_{1 \le i \le N}$ is an indexed family of submodules of V, then V is the internal direct sum of U_i iff the rule $\{i_0, \ldots, i_m\} \mapsto U_{i_0} + \cdots + U_{i_m}$ defines a complete lattice homomorphism

$$\Lambda : \mathbb{P}(\{1, \ldots, N\}) \to \mathrm{Sub}(V).$$

If this is not immediately clear, the example $\mathbf{R}^2 \cong \mathbf{R} \oplus \mathbf{R}$ may prove a useful exercise. This connection places finite internal direct sums in canonical 1-1 correspondence with bounded lattice homomorphisms indexed by $\mathbb{P}(P)$.

A.2 Homological persistence: $P = \mathbf{R}^2$

In the case $P = \{1, \ldots, N\}$ we saw that direct sums correspond to lattice homomorphisms $\mathbb{P}(P) \to \mathrm{Sub}(V)$, while filtrations correspond to homomorphisms $\mathbb{A}(P) \to \mathrm{Sub}(V)$. These decomposition schemes have unique strengths and weaknesses that can be combined and exploited. These principles hold for lattice homomorphisms indexed by $\mathbb{P}(P)$ and $\mathbb{A}(P)$ generally. Let us see how they manifest in the context of homological persistence, with $P = \mathbf{R}^2$.

For concreteness, fix a ground field $\Bbbk$ and a diagram $f : \mathbf{R} \to \Bbbk\text{-}\mathtt{Mod}$ in the abelian category $[\mathbf{R}, \Bbbk\text{-}\mathtt{Mod}]$ of $\mathbf{R}$-shaped diagrams in $\Bbbk\text{-}\mathtt{Mod}$ and natural transformations between them. Suppose that $f = g_1 \boxplus \cdots \boxplus g_4$ is the internal direct sum of an indexed family of subdiagrams g such that each g_s is an interval module supported on $[s, s+2)$. Figure 4 displays the associated barcode and persistence diagram. To these data one can associate lattice homoorphisms

$$\Lambda^g : \mathbb{P}(\mathbf{R}^2) \to \mathrm{Sub}(f) \qquad T \mapsto \mathrm{span}(\{g_s : (s, s+2) \in T\}).$$

and

$$\Lambda = \Lambda^g|_{\mathbb{A}(T)} : \mathbb{A}(\mathbf{R}^2) \to \mathrm{Sub}(f)$$

As outline below, these homomorphisms correspond to direct sum and *bifiltration*.

A.2.1 Direct sum

We noted earlier a 1-1 correspondence between internal direct sum decompositions of a module V into an indexed family of N summands on the one hand, and lattice homomorphisms $\mathbb{P}(P = \{1,\dots,N\}) \to \mathrm{Sub}(V)$ on the other. One can of course substitute $\{1,\dots,N\}$ for any poset of cardinality N and obtain the same result. One can likewise substitute V for any object in an Abelian category, though in this case one must also declare what is meant by *internal*, eg, by fixing a universe. Homomorphism Λ^g is the canonical lattice map associated to g, under this correspondence.

This map has several attractive properties. Given an oracle that returns the value of Λ^g on any element of $\mathbb{P}(P)$, for example, the barcode and persistence diagram of f become readily computable, since, in particular, $\Lambda^g\{(a,b)\}$ is the sum of all g_s supported on $[a,b)$.

However, there is something unnatural about Λ^g. It is canonically determined by g, and *not* by f. For example, one can construct an internal direct sum decomposition $f = \boxplus_{s\in S} g'_s$ such that $\Lambda^g \neq \Lambda^{g'}$.

This presents a problem when f is the basic object of interest. To formulate the Krull-Schmidt decomposition of f, which lies at the heart of most modern treatments of homological persistence, one must "add" structure. In many cases this pollution leads to no difficulty in particular, but in many others the problem of disentangling what is essential to f from what is essential to g becomes entrenched. This is particularly the case in homological algebra, where naturality is key. This example typifies a general confound of direct sums: like a choice of basis, they are often useful, but seldom natural.

A.2.2 Bifiltration

In a previous example we noted that (bounded, finite) filtrations, or lattice homomorphisms $\mathbb{A}(P = \{1,\dots,N\}) \to \mathrm{Sub}(V)$, appear more often in nature than do direct sums, thanks to lighter structure requirements. By the same token, these maps also tend to behave in a natural manner, categorically.

Such is the case in the persistent context, also. While $\Lambda^g : \mathbb{P}(\mathbf{R}^2) \to \mathrm{Sub}(f)$ is not uniquely determined by f, the restriction $\Lambda : \mathbb{A}(\mathbf{R}^2) \to \mathrm{Sub}(f)$ is. The proof follows from the fact that $\mathbb{A}(\mathbf{R}^2)$ is a complete sublattice of $\mathbb{P}(\mathbf{R}^2)$, that Λ^g is a complete lattice homomorphism, and that Λ^g *is* uniquely determined on sets of form $(-\infty,t] \times \mathbf{R}$ and $\mathbf{R} \times (-\infty,t]$, as we will show.

Given direct access to the values of Λ, one can compute the barcode of f, and therefore its isomorphism type, with relative ease:

$$\Lambda(\downarrow(b,d)) / \Lambda(\mathring{\downarrow}(b,d)) \cong \mathrm{span}(g_s : \mathrm{birth}(g_s) = b,\ \mathrm{death}(g_s) = d). \qquad (22)$$

This requires a bit more work than would be strictly necessary if we had direct access to Λ^g: with the latter, one only has to evaluate $\Lambda^g_{\{(b,d)\}}$ and count the dimension of the stalks; with Λ one has to evaluate both $\Lambda_{\downarrow(b,d)}$ and $\Lambda_{\mathring{\downarrow}(b,d)}$, count dimension, and take a difference.

We see in this example the same interplay between $\mathbb{P}$ and $\mathbb{A}$ as remarked in the discussion of direct sums and filtrations. Lattice homomorphism Λ^g is defined on a $\mathbb{P}(\mathbf{R}^2)$, and corresponds to a biproduct. It is simple to study and simple to evaluate: one merely collects the submodules whose birth/death pairs lie in T.

However, there is something unnatural about Λ^g. It is not canonically defined, has no closed form expression in f.

On the other hand, homomorphism Λ *is* canonically defined, and more natural. It has more of the flavor of a filtration than a direct sum. In fact we will see that it is uniquely determined by the filtrations $\Lambda \circ \rho_0$ and $\Lambda \circ \rho_1$ defined

$$\rho_0(t) = \mathbf{R}_{\leq t} \times \mathbf{R} \qquad\qquad \rho_1(t) = \mathbf{R} \times \mathbf{R}_{\leq t},$$

so the comparison is quite close.

Since Λ is uniquely determined by f, every detail of its structure relates directly to the problem of interest. The price of this purity is comfort and familiarity. The power set lattice is more common to the vernacular than the Alexandrov topology. Computations, whether by hand or machine, are simpler and more intuitive in regimes where biproducts abound. This is evidenced by the two-decade gap between the date of the first definition of the persistence diagram for modules with linear coefficients by Robinson in 19xx, and that of Abelian modules by Patel in 20xx. Today, the former has over 30 computer implementations deployed across research and industrial organizations in mathematics, engineering, and the sciences. At present, the latter has none.

A.2.3 The general case

The example of Λ and Λ^g strongly suggests a key structural relationship between the persistence module f and the family of homomorphisms $\mathbb{X}(P) \to \mathrm{Sub}(f)$, where $\mathbb{X} \in \{\mathbb{A}, \mathbb{P}\}$. In this example, poset P is the product order $\mathbf{R} \times \mathbf{R}$, the first factor of which corresponds, conceptually, to birth time, the second to death. This choice of P serves well for a variety of problems, but a general persistence module can be any functor

$$f : \mathbf{I} \to E$$

where $\mathbf{I}$ is a total order. Common choices for $\mathbf{I}$ include $\mathbf{Z}$, $\mathbf{Z}_{\geq 0}$, and $\{0, \ldots, N\}$ in addition to $\mathbf{R}$, so it behooves us to consider the general case $P = \mathbf{I} \times \mathbf{I}$. Having determined to operate on this level of generality, it will add little to overall complexity (and will in fact add much to conceptual and notational clarity) to extend our scope to posets $P = \mathbf{I} \times \mathbf{J}$, where $\mathbf{J}$ is any total order. This product will lie at the foundation of much of our work.

As to the target E, this may be any category, in principle. To have meaningful notions of birth, death, etc., however, one generally wants formal notions of image and kernel, hence exactness. We therefore assume that E is one of the following: (i) a category of vector spaces and linear maps, (ii) an abelian category, or (iii) a Puppe exact category. The reader may assume vector spaces at loss to essential understanding.

In summary, we have trained our attention on lattice homomorphisms

$$\Lambda : \mathbb{X}(\mathbf{I} \times \mathbf{J}) \to \mathrm{Sub}(V) \qquad\qquad (\mathbb{X} \in \{\mathbb{P}, \mathbb{A}\})$$

where $\mathbf{I}$ and $\mathbf{J}$ are total orders, and where V is some object in a category of the reader's choosing. These maps constitute the basic object of our study.

B Appendix: continuous and algebraic lattices

If a, b are elements of a complete lattice $\mathbf{L}$, then a is *way below* b, written $a \ll b$, if for every S such that $b \leq \bigvee S$ there exists a finite $T \subseteq S$ such that $a \leq \bigvee T$. Element a is *compact* if $a \ll a$.

A *base* for $\mathbf{L}$ is a subset $B \subseteq \mathbf{L}$ such that every $b \in \mathbf{L}$ is the supremum of some directed set $D \subseteq B$. A base is *compact* if every element of B is compact.

A complete lattice is *continuous* if it satisfies any of the following equivalent criteria: (i) $\mathbf{L}$ has a base, (ii) $B = \{b \in \mathbf{L} : b \ll c \text{ for some } c\}$ is a base, (iii) $c = \bigvee_{b \ll c} b$ for all $c \in \mathbf{L}$. A continuous lattice is *algebraic* if it has a compact base.

Example 4. *Let V be a module of ring R, and $\mathbf{L}$ be the lattice of submodules of V. Then every cyclic submodule is compact as an element of $\mathbf{L}$. The compact elements of $\mathbf{L}$ are exactly the finite joins of cyclic submodules, and the set of all such modules is a base for $\mathbf{L}$. In particular, $\mathbf{L}$ is continuous.*

Example 5. *Let $\mathbf{L} = \boldsymbol{R} \cup \{-\infty, \infty\}$ be the extended real line. Then $\mathbf{L}$ is a base unto itself, and this base is not compact.*

A lattice $\mathbf{L}$ is *upper continuous* or *meet continuous* iff for each $x \in \mathbf{L}$, each directed set D, and each poset homomorphism $a : D \to \mathbf{L}$ one has $x \wedge (\bigvee a) = \bigvee_d (x \wedge a_d)$. It is *lower continuous* or *join continuous* iff $\mathbf{L}^*$ is meet continuous. It is a well known fact that a complete lattice is *completely distributive* iff it is both upper and lower continuous. By another useful criterion [18], a complete lattice $\mathbf{L}$ is completely distributive iff there exists a surjective complete lattice homomorphism $\mathbf{M} \to \mathbf{L}$ for some complete ring of sets $\mathbf{M}$.

Lemma 22. *Every lattice with a compact base is meet-continuous.*

Proof. Let $\mathbf{L}$ be a lattice with base B. Let D is a directed set and $a : D \to \mathbf{L}$ a poset homomorphism. If $x = \bigvee_{d \in D} a_d$ and $b \in B_{\leq x}$ then there exists a finite subset $S \subseteq D$ such that $b \leq \bigvee_{s \in S} a_s$, by compactness. Since D is directed, there then exists $d \in D$ such that $b \leq a_d$. Thus $B_{\leq x} = \bigcup_{d \in D} B_{\leq a_d}$. Now, fix $c \in \mathbf{L}$, and suppose that $y = \bigvee_{d \in D} (a_d \wedge c)$ exists. Then

$$B_{\leq y} = \bigcup_{d \in D} B_{\leq a_d \wedge c} = \bigcup_{d \in D} (B_{\leq a_d} \cap B_{\leq c}) = B_{\leq x} \cap B_{\leq c} = B_{\leq x \wedge c}$$

hence $y = \bigvee B_{\leq y} = \bigvee B_{\leq x \wedge c} = x \wedge c$. The desired conclusion follows. $\square$

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