Invitation to Matroid Theory

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Abstract

This text is a companion to the short course *Invitation to matroid theory* taught in the University of Oxford Centre for TDA in January 2021. It was first written for algebraic topologists, but should be suitable for all audiences; no background in matroid theory is assumed!

1 How to use this text

Matroid theory is a beautiful, powerful, and accessible. Many important subjects can be understood and even researched with just a handful of concepts and definitions from matroid theory.

Even so, getting use out of matroids often depends on knowing the right keywords for an internet search, and finding the right keyword depends on two things: (i) a precise knowledge of the core terms/definitions of matroid theory, and (ii) a birds eye view of the network of relationships that interconnect some of the main ideas in the field. These are the focus of this text; it falls far short of a comprehensive overview, but provides a strong start to (i) and enough of (ii) to build an interesting (and fun!) network of ideas including some of the most important concepts in matroid theory.

The proofs of most results can be found in standard textbooks (e.g. Oxley, *Matroid theory*). Exercises are chosen to maximize the fraction

$$\frac{\text{intellectual reward}}{\text{time required to solve}}$$

Therefore, *if an exercise takes more than a few minutes to solve, it is fine to look up answers in a textbook! Struggling is not prerequisite to learning, at this level.*

2 References

The following make no attempt at completeness. In fact, they leave most fields of matroid theory untouched, e.g. oriented matroids, applications in mechanism design, combinatorial convex geometry, etc.

1. An encyclopedic reference
   - Oxley, *Matroid theory*
2. A (non-encyclopedic but highly) popular introduction
   - Reiner, *Lectures on matroids and oriented matroids*
3. An article-length introduction

- Neel and Neudauer, *Matroids you have known*

4. Matroids in algebraic geometry
    - Eric Katz, *Matroid theory for algebraic geometers*
5. Geometry of matroids
    - Federico Ardila, *Geometry of matroids*
6. Category theory of matroids
    - Heunen, Chris, and Vaia Patta. *The category of matroids.* Applied Categorical Structures 26.2 (2018): 205-237.
7. Minors and decomposition of matroids
    - Truemper, *Matroid Decomposition*
8. Homology of matroids
    - Björner, *Homology and shellability of matroids and geometric lattices*
    - See also matroids in algebraic geometry
9. Homotopy of matroids
    - Tutte, *A homotopy theorem for matroids* (the homotopy theorem was used to prove Tutte's forbidden-minor characterization of binary matroids, a landmark result)
    - See also matroids in algebraic geometry
10. Matroids in TDA
    - Chen, Chao, and Daniel Freedman. *Measuring and computing natural generators for homology groups.* Computational Geometry 43.2 (2010): 169-181.
        - Uses matroid structure to find optimal cycles.
    - Erickson, Jeff, and Whittlesey, Kim. *Greedy optimal homotopy and homology generators.* SODA. Vol. 5. 2005.
        - a popular article - 244 citations
    - Farber, Michael, Serge Tabachnikov, and Sergey Yuzvinsky. *Topological robotics: motion planning in projective spaces.* International Mathematics Research Notices 2003.34 (2003): 1853-1870. APA
        - Authors use the Orlik-Solomon structure of complex hyperplane arrangements, a structure predicated on matroids
    - Henselman, Gregory, and Robert Ghrist. *Matroid filtrations and computational persistent homology.* arXiv preprint arXiv:1606.00199 (2017).
        - Authors show matroid structure of persistent homology cycle bases and use this to probe fill patterns of sparse boundary matrices.
    - Henselman-Petrusek, Gregory. *Matroids and Canonical Forms: Theory and Applications.* arXiv preprint arXiv:1710.06084 (2017).
        - Detailed discussion of selected topics in matroids, persistence, algorithms, and related structures e.g. discrete Morse theory.
    - Henselman, Gregory, and Pawel Dłotko. *Combinatorial invariants of multidimensional topological network data.* 2014 IEEE Global Conference on Signal and Information Processing (GlobalSIP). IEEE, 2014.

  - Sampling of topics in TDA, including an application of max-flow min-cut matroids to robustness of coverage in sensor networks.

- Skraba, Primoz, Gugan Thoppe, and D. Yogeshwaran. *Randomly Weighted $d-$ complexes: Minimal Spanning Acycles and Persistence Diagrams.* arXiv preprint arXiv:1701.00239 (2017).
  - Fascinating discussion of cellular matroids for randomly weighted complexes.

3 Notation and conventions

We write $\langle S \rangle$ for the subspace of a vector space V generated by a subset $S \subseteq V$.

Where context leaves no room for confusion we omit vertices from graphs. In general, these may be inferred from the transverse intersection of line segments. Thus Figure 1 shows a (wheel) graph with 4 vertices and 6 edges on the left, and a graph with 6 vertices and 12 edges on the right.

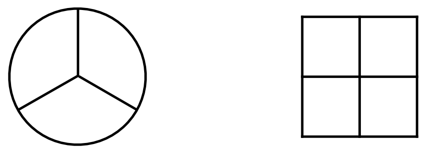

Figure 1: *Left* the wheel graph with 4 vertices and 6 edges. *Right* a graph with 6 vertices and 12 edges.

4 Why study matroids?

Reiner 2005*: Here are a few of my personal reasons.*

1. *The are general, so results about them are widely applicable*
2. *The have relatively few axioms and standard constructions/techniques, so they focus one's approach to solving a problem.*
3. *They give examples of well-behaved objects: polytopes, cell/simplicial complexes, rings*
4. *They provide "duals" for non-planar graphs!*

5 What is a matroid? Definition by independent sets

Matroids have many equivalent formulations. However, the overwhelming majority of papers that use matroids only rely on a 1 to 3 different formulations. Therefore a winning strategy for the student or casual user is to (i) learn one or two of the standard definitions, and (ii) fill in gaps as needed, on the fly.

Matroid characterization 1 (by independence). *A* (finite rank, finitary) matroid *is a pair* $\mathbf{M} = (E, \mathbf{I})$*, where E is a set, $\mathbf{I}$ is a nonempty family of bounded-cardinality (meaning $< N$, for some N) subsets of E, and*

1. $\mathbf{I}$ *is closed under taking subsets*
2. $|A| < |B|$ *for some* $A, B \in \mathbf{I} \implies \exists b \in B \backslash A$ *such that* $A \cup \{b\} \in \mathbf{I}$.

Definition 1 (Ground set)**.** *The* ground set *of* $\mathbf{M}$ *is E. As per convention, we write*

$$E(\mathbf{M}) := E.$$

Definition 2 ((In)dependent sets)**.** *Elements of* $\mathbf{I}$ *are called* independent. *Elements of* $\mathbf{I}^c := 2^E - \mathbf{I}$ *are called* dependent. *As per convention,*

$$\mathbf{I}(\mathbf{M}) := \mathbf{I}.$$

Exercise 1. *Confirm that each of the following defines a matroid.*

1. **Matroid of a vector configuration** (linear matroid)*:* $E(\mathbf{M})$ *is a set,* $\phi : E(\mathbf{M}) \to V$ *is a finite dimensional vector configuration, and* $I \in \mathbf{I}$ *iff* $\phi(I)$ *spans a subspace of dimension* $|I|$.

   Remark (etymology of the term matroid) To each matrix $A \in \mathbb{K}^{m \times E}$ *corresponds a function* $\phi : E \to \mathbb{K}^m$, $e \mapsto A[:,e]$. *Thus every matrix engenders a matroid.*

2. **Matroid of a graph** *(graphic matroid)* $E(\mathbf{M})$ *is the family of 1-cells in a CW complex* X*, and* $\mathbf{I}(\mathbf{M})$ *is the family of forests in* E.

   Recall a forest is a set of edges admitting no nonempty closed path.

   Hint edges are in bijection with columns of the boundary matrix ∂_1.

3. **Matroid of a CW complex** (cellular matroid) $\mathbb{K}$ *is a field,* X *is a CW complex,* $E(\mathbf{M})$ *is the family of n-cells of* X*, and* $\mathbf{I}(\mathbf{M})$ *is the family of all sets of n-cells* I *such that matrix* $\partial_n[:,I]$ *has full column rank.*

   Remark Elements of $\mathbf{I}(\mathbf{M})$ *are n-dimensional cellular* $\mathbb{K}$*-forests of* X.

   For reflection the term forest *is underspecified for the matroid of n-cells, in general; one must therefore speak of* $\mathbb{K}$*-forests. Why, then, are fields unmentioned when referring to the forests of a graph?*

4. **Matroid of a geometric order lattice** *(simple matroid)* L *is a geometric (meaning finite, atomistic, and semimodular) order lattice.* $E(\mathbf{M})$ *is the set of atoms of* L*, and* $I \in \mathbf{I}(\mathbf{M})$ *iff the join* $\vee I$ *is irredundant.*

5.1 Definition by bases

Definition 3. *A* basis *of* $\mathbf{M}$ *is a maximal (with respect to inclusion) independent set. The set of bases is denoted*

$$\mathbf{B}(\mathbf{M}) := \max(\mathbf{I}(\mathbf{M})).$$

Example 1 (cf Reiner 2005)**.**

1. ***Linear matroids*** *induced by a vector configuration* $\phi : E \to V$
   $\mathbf{I}(\mathbf{M}) =$ *(sets mapping injectively into) linearly independent subsets*
   $\mathbf{B}(\mathbf{M}) =$ *(sets mapping bijectively onto) bases of the span of* $\phi(E)$

2. ***Algebraic matroids*** *(S. Mac Lane 1938)) induced by a function* $\phi : E \to \mathbb{K}'$*, where* $\mathbb{K}'$ *is an extension field of* $\mathbb{K}$.
   $\mathbf{I}(\mathbf{M}) =$ *(sets mapping injectively into) algebraically ind. subsets*
   $\mathbf{B}(\mathbf{M}) =$ *(sets mapping bijectively onto) transcendence bases of the generated subfield*

3. ***Graphic matroids*** *induced by an undirected graph (realized as a CW complex) G.*
   $\mathbf{I}(\mathbf{M})=$ *forests of edges*
   $\mathbf{B}(\mathbf{M})=$ *spanning forests (spanning trees, if G is connected)*

4. ***Transversal matroids*** *(J. Edmonds and D.R. Fulkerson 1965) induced by a bipartite graph on edge set (E,F), where $E\cap F=\emptyset$.*
   $\mathbf{I}(\mathbf{M})=$ *endpoints in E of partial matchings*
   $\mathbf{B}(\mathbf{M})=$ *endpoints in E of max-cardinality matchings*

Matroid characterization 2 (by bases). *A family $\mathbf{B}\subseteq 2^E$ is the family of bases of a matroid on ground set E iff for all $B,C\in\mathbf{B}$ and $b\in B-C$ there exists $c\in C$ such that $(B-\{b\})\cup\{c\}\in\mathbf{B}$.*

5.2 Definition by rank

Structure	E	**I**	max(**I**)
matrix	column vectors	linearly independent subsets	column basis
undirected multigraph	edges	forests	spanning forest
geometric order lattice	atoms	irredundant sets	basis
bipartite graph, edges $A\subseteq E\sqcup F$	vertex set E	E-endpoints of partial matchings	maximal matched set
field extension K/k	elements of K	alg. ind. subsets	transcendence basis

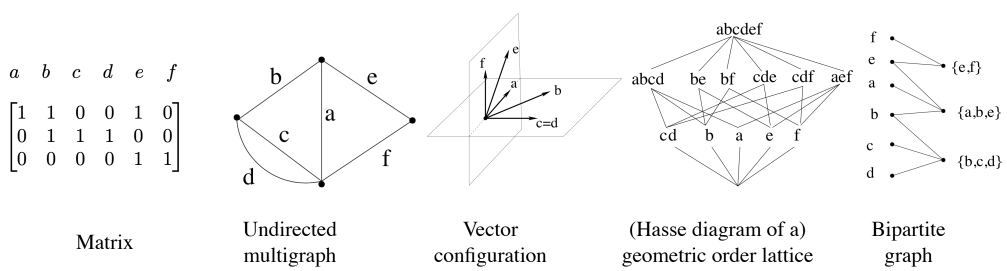

Figure 2: Each object engenders the same matroid (with the exception of the geometric lattice, which identifies c and d). *Figure credit: Reiner, Lecture notes on matroids and oriented matroids*

Exercise 2 (Challenge problem - for those with spare time)**.** *Prove that each object in Figure 2 (except for the geometric lattice) engenders the same matroid.*

Definition 4 (Rank function of a matroid)**.** *The* rank function *of* $\mathbf{M}$ *is*

$$\rho_{\mathbf{M}}:2^E\to\mathbb{R}\qquad\qquad \rho_{\mathbf{M}}(S):=\max\{|I|: I\subseteq S \text{ is independent }\}.$$

We say S has rank $\rho_{\mathbf{M}}(S)$.

Exercise 3.

1. *Show that if* $\mathbf{M}$ *is the matroid of linearly independent subsets of* $E \subseteq V$*, then*

$$\rho_{\mathbf{M}}(S) = \dim\langle S\rangle$$

for all $S \subseteq E$.

2. *Verify the following when* $\mathbf{M}$ *is the matroid represented by (i) the multigraph, (ii) the vector configuration, and (iii) the bipartite graph in Figure 2, using the definitions of those matroids:*

$$\rho_{\mathbf{M}}(\{c,d\}) = 1 \qquad \rho_{\mathbf{M}}(\{b,c,d\}) = 2 \qquad \rho(E) = 3.$$

Definition 5 (Submodular function)**.** *A set function* $\rho : 2^E \to \mathbb{R}$ *is* submodular *if*

$$\rho(S\cap T) + \rho(S\cup T) \leq \rho(S) + \rho(T) \tag{1}$$

for all $S,T \subseteq E$*. We call* (1) *the* submodular inequality.

Etymology *The term* submodular *relates to the idea of a* modular pair *in lattice theory. It turns out that if* S *and* T *are* flats *of* $\mathbf{M}$ *(see definition below), then* (S,T) *is a modular pair iff* (1) *holds with strict equality (c.f. Oxley,* Matroid theory*).*

Matroid characterization 3 (by rank). *A function* $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ *is the rank function of a matroid on ground set* E *iff* ρ *is submodular and*

$$0 \leq \rho(S) \leq \rho(T) \leq |T| \tag{2}$$

for all $S \subseteq T \subseteq E(\mathbf{M})$.

Exercise 4. *Suppose* $\mathbf{M}$ *is the (finite rank) matroid of linearly independent subsets of* $E \subseteq V$.

1. *Check the submodular inequality* (1) *and inequality* (2).
   Hint: If $S,T \subseteq E$ *and* $U := \langle S\rangle$, $V := \langle T\rangle$*, then*

$$\underbrace{\dim(U\cap V)}_{\geq\rho(S\cap T)} + \underbrace{\dim(U+V)}_{\rho(S\cup T)} = \underbrace{\dim(U)}_{\rho(S)} + \underbrace{\dim(V)}_{\rho(T)}.$$

2. *Persuade yourself that the submodular inequality can hold with strict inequality for certain matroids* $\mathbf{M}$.
   Aside It turns out that most matroids are subject to failure of strict equality for certain choices of S *and* T*; in fact, strict equality for all* S *and* T *implies that, up to removal of "duplicate" elements,* $\mathbf{M}$ *is a union of free matroids and projective spaces, c.f. Oxley,* Matroid theory.

5.3 Cryptomorphisms

One has

$$
\begin{aligned}
I \in \mathbf{I}(\mathbf{M}) &\iff \rho_{\mathbf{M}}(I) = |I| \\
&\iff I \subseteq B \text{ for some } B \in \mathbf{B}(\mathbf{M}) \\
B \in \mathbf{B}(\mathbf{M}) &\iff |B| = \rho_{\mathbf{M}}(B) = \rho_{\mathbf{M}}(E) \\
&\iff B \in \max \mathbf{I} \\
\rho_{\mathbf{M}}(S) = m &\iff m = \max\{|I| : I \subseteq S,\ I \in \mathbf{I}(\mathbf{M})\} \\
&\iff m = \max\{|I| : I \subseteq S \cap B \text{ for some } B \in \mathbf{B}(\mathbf{M})\}
\end{aligned}
$$

Thus we have canonical transformations

$$
\begin{tikzcd}
& \text{pairs } (E,\mathbf{I}) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{pairs } (E,\mathbf{B}) \arrow[rr, leftrightarrow] & & \text{pairs } (E,\rho)
\end{tikzcd}
\tag{3}
$$

Definition 6 (Cryptomorphism)**.** *The arrows in* (3) *are neither isomorphisms nor functors, though they are akin to both. Matroid theorists call them* cryptomorphisms. *There is no formal definition of a cryptomorphism; "you just know it when you see it."*

There are many, many cryptomorphisms in matroid theory.

5.4 More on cryptomorphisms

Many cryptomorphisms are simple in nature and handy for everyday use. There are also quite powerful and lovely cryptomorphisms throughout mathematics. See, for example

1. *Combinatorial Geometries, Convex Polyhedra, and Schubert Cells* by Gelfand, Goresky, macPherson, Serganova
2. *Matroid theory for algebraic geometers* by Eric Katz
3. *Matroid theory* by James Oxley

5.4.1 Projective geometry (optional reading)

"*[Projective geometries] are extremely natural to consider in matroid theory, their position among representable matroids being analogous to that of complete graphs in graph theory.*" - Oxley, *Matroid theory*

Reference 1. *For an introduction to projective geometries in matroid theory, see Oxley,* Matroid theory*, starting p. 163.*

Definition 1 (Projective space)**.** *A* projective space *is a triple (P, L, ι) (interpreted as points, lines, and incidence relation, respectively) such that $P \cap L = \emptyset$, and*

1. *Any two distinct points $a, b \in P$, are incident to exactly one line $ab \in L$.*

2. *Every line contains at least 3 points.*

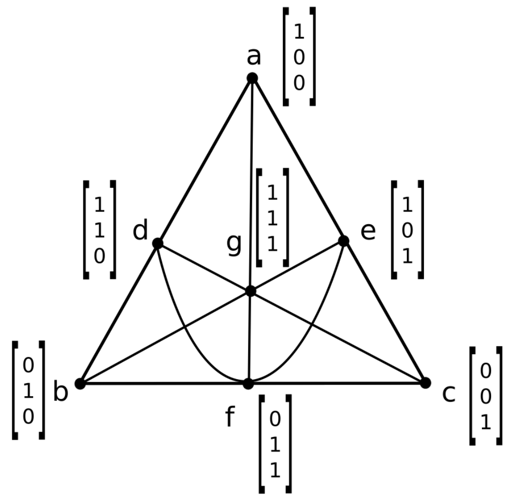

Figure 3: The Fano plane $F_7 = PG(2,2)$, from Neel and Neudauer, *Matroids you have known*. F_7 is the smallest projective plane.

3. *If a,b,c,d are four distinct points, no three of which are collinear, and if the line ab intersects the line cd, then the line ac intersects the line bd.*

Definition 2 (Projective geometry)**.** *Let V be a (finite dimensional) vector space over a finite field $\mathbb{K}$. The* projective geometry *of V is the triple (P, L, ι), where P and L are the sets of 1- and 2-dimensional subspaces (that is, projective points and lines), respectively, and ι is the incidence relation between P and L.*

Fact Every projective geometry is a projective space.

6 Circuits, flats, closure, examples

Definition 7 (Circuits)**.** *A circuit is a minimal (with respect to inclusion) dependent set. The family of circuits is denoted*

$$\mathbf{C}(\mathbf{M}) := \min(2^{E(\mathbf{M})} - \mathbf{I}(\mathbf{M}))$$

Variation in meaning across fields *in graph theory,* circuit *refers to any closed trail, while a* cycle *or* simple circuit *refers to a circuit without repeat vertices. This turns out to differ from the matroid theoretic use of the term.*

Exercise 1. *If $\mathbf{M}$ is the matroid of an undirected multigraph G, then the circuits of $\mathbf{M}$ coincide exactly with the nonempty cycles (simple closed paths) of G.*

Matroid characterization 4 (by circuits). *If E is finite, then a family of subsets $\mathbf{C} \subseteq 2^E$ form the circuits of a matroid $\mathbf{M}$ iff*

1. $\emptyset \notin \mathbf{C}$
2. $\zeta_1, \zeta_2 \in \mathbf{C} \wedge \zeta_1 \subseteq \zeta_2 \implies \zeta_1 = \zeta_2$
3. $\zeta_1, \zeta_2 \in \mathbf{C} \wedge \zeta_1 \neq \zeta_2 \wedge e \in \zeta_1 \cap \zeta_2 \implies \exists \zeta_3 \in \mathbf{C}$ *such that* $\zeta_3 \subseteq (\zeta_1 \cup \zeta_2) - \{e\}$.

Definition 8 (Closure)**.** *The* closure *of $S \subseteq E$ is*

$$\mathrm{cl}(S) := \{e \in E : \rho(S \cup \{e\}) = \rho(S)\}.$$

Matroid characterization 5 (by closure operator)**.** *If E is a finite set, then* $\mathrm{cl} : 2^E \to 2^E$ *is the closure operator of a matroid iff (i)* cl *is a closure operator (monotone, extensive, idempotent) and (ii)* $S \subseteq E \wedge s \in E \wedge t \in \mathrm{cl}(S \cup \{s\}) - (S) \implies x \in \mathrm{cl}(\mathrm{S} \cup \{\mathrm{t}\})$

Definition 9 (Flats)**.** *A* flat *of* $\mathbf{M}$ *is a subset $F \subseteq E$ such that* $\mathrm{cl}(F) = F$. *The set of all flats forms*

$$\mathcal{F}(\mathbf{M}) := \{F \subseteq E : F = \mathrm{cl}(F)\}.$$

Theorem 1 (Order lattice of flats)**.** *The family of flats $\mathcal{F}(\mathbf{M})$ forms an order lattice under inclusion; specifically,*

$$F \wedge G = F \cap G \qquad\qquad F \vee G = \mathrm{cl}(F \cup G)$$

for all $F, G \in \mathcal{F}(\mathbf{M})$.

Exercise 5. *Prove Theorem 1.*

Matroid characterization 6 (by flats). *A lattice L is geometric (that is, finite, atomistic, and semimodular) iff it is isomorphic to the lattice of flats of a finite matroid.*

Matroid characterization 7 (by flats (again)). *A subset $\mathcal{F} \subseteq 2^E$ forms the lattice of flats of a matroid on ground set E iff*

1. $E \in \mathcal{F}$
2. $F_1, F_2 \in \mathcal{F} \implies F_1 \cap F_2 \in \mathcal{F}$
3. *If $F \in \mathcal{F}$ and* $\{F_1, \ldots, F_k\} = \min\{G \in \mathcal{F} : F \subsetneq G\}$, *then* $E - F = F_1 - F \sqcup \cdots \sqcup F_k - F$.

Exercise 6. *Verify Table 1.*

Exercise 7. *Figure 2 displays seven sets of edges contained in several undirected multigraphs, labeled A-G. Verify the properties of these sets as recorded in Table 2.*

Table 1: Properties of subsets of $\mathbf{M}$

Object	Rank function	Independent Sets	Injective Vec. Config.	Graph
Ind. set I	$\rho(I) = \lvert I \rvert$	$I \in \mathbf{I}$	$\phi(I)$ is lin. indep.	forest
Basis B	$\rho(B) =$ $\lvert B \rvert = \rho(E)$	$B \in \max(\mathbf{I})$	$\phi(B)$ is a basis for $\langle \phi(E) \rangle$	spanning forest
Circuit C	$\rho(S) =$ $\min(\lvert S \rvert, \lvert C \rvert - 1)$ $\forall S \subseteq C$	$S \subseteq C \implies$ $(S \in \mathbf{I} \iff S \neq C)$	$S \subseteq C \implies \phi(S)$ is lin. ind. iff $S \neq C$	cycle
Flat F	$\rho(F) < \rho(S)$ $\forall F \subsetneq S$	$I \cup \{e\} \in \mathbf{I}$ $\forall e \notin F, F \supseteq I \in \mathbf{I}$	$\phi(F) =$ $\phi(E) \cap \langle \phi(F) \rangle$	$\exists$ path $x \rightsquigarrow y$ in F $\implies F$ contains all length-1 paths $x \rightsquigarrow y$.

Table 2: Properties of edge sets

Property	A	B	C	D	E	F	G
Independent	X		X	X			
Basis			X	X			
Circuit		X			X		
Flat	X	X				X	

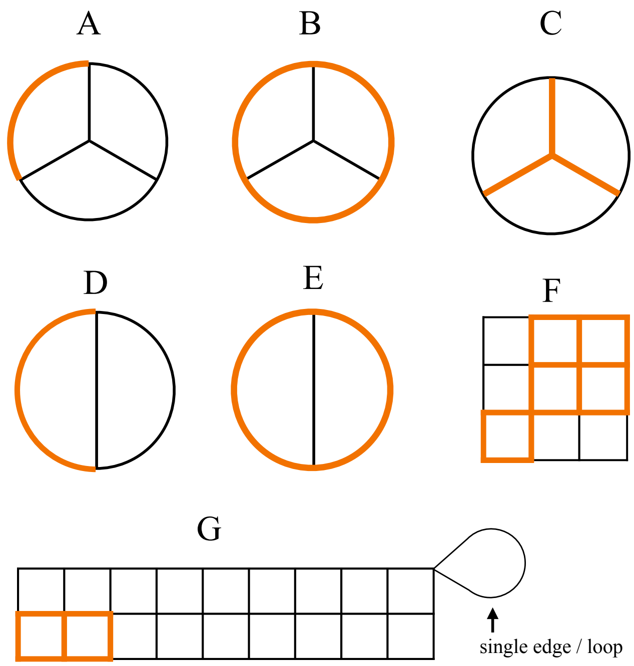

Figure 4: Edge sets

7 Linear representations

A *representation* of matroid $\mathbf{M}$ is a mathematical object that engenders $\mathbf{M}$ (such as a vector configuration). There are many types of representations (linear, algebraic, topological, graphic, etc.). *Unless otherwise stated* an author using the term "representable" means "linearly representable."

Definition 10 (Linear representation, linearly reprsentable)**.** *A* $\mathbb{K}$-linear representation *of* $\mathbf{M}$ *is a vector configuration* $\phi : E(\mathbf{M}) \to V$ *such that* $I \in \mathbf{I}(\mathbf{M})$ *iff* $\phi(I)$ *spans a subspace of dimension* $|I|$.

A matroid is $\mathbb{K}$-representable *if it admits a* $\mathbb{K}$*-linear representation, and* representable *if it admits a* $\mathbb{K}$*-linear representation for some* $\mathbb{K}$*.*

Theorem 2. *The problem of determining whether a matroid is representable over a specific field* $\mathbb{K}$ *is NP. However, the problem of determining whether a matroid is representable over* all *fields is polynomial (Truemper, 1982).*

Remark 1. *Matroids representable over every field are called* regular*. Regular matroids have extremely strong structural properties, and furnished the first polynomial-time algorithm to test matrices for total unimodularity.*

8 Day 2

Recall:

1. Definition of *closure*, *flat*
2. Remark: the family of flats $\mathcal{F}(\mathbf{M})$ forms an order lattice under inclusion. (Exercise: prove this)
3. The following can also be axiomatized to define a matroid: circuits, flats, closure operators
4. Review of Table 1

9 Matrix representations

Matrix representations play a fundamental role in the study of matroids. Matroid theorists make an art of drawing deep structural properties to the surface by means of such representations. The *standard form* is a fundamental tool in this toolset.

Definition 3 (Nonstandard matrix representation)**.** *A* nonstandard matrix representation *of* $\mathbf{M}$ *is a matrix* $M \in \mathbb{K}^{F \times E}$ *such that the vector configuration* $\phi : E(\mathbf{M}) \to \mathbb{K}^F,\ e \mapsto M[:,e]$ *represents* $\mathbf{M}$.

Definition 11 (Standard matrix representation)**.** *Suppose that* $E(\mathbf{M}) = X \sqcup Y$*, where* X *is a basis of* $\mathbf{M}$*. A* standard form *matrix representation of* $\mathbf{M}$ *with respect to* X *is a matrix* $B \in \mathbb{K}^{X \times Y}$ *such that*

$$A = \begin{array}{c|c|c|} & X & Y \\ \hline X & I & B \\ \hline \end{array}$$

is a nonstandard *matrix representation of* $\mathbf{M}$*, where* I *denotes the dirac delta function on* X*.*

Exercise 2. *Verify that every non-standard matrix representation* F *can be refined to a standard matrix representation* B *by a process analogous to that shown in Figure 5.*

The standard form has several key advantages:

1. Compactness - deleting $I \in \mathbb{K}^{X \times X}$ removes redundant information.
2. Duality - standard forms play very well with dual matroids, as we will see.
3. Fundamental circuits - stand forms trivialize the task of reading fundamental circuits (Exercise 4).

Remark 1. *The exchange of row and column labels when deriving a B-standard representation relates naturally to the reversal of arrows in a discrete Morse vector field on a CW complex (H.P.,* Matroids and Canonical forms: Theory and applications*).*

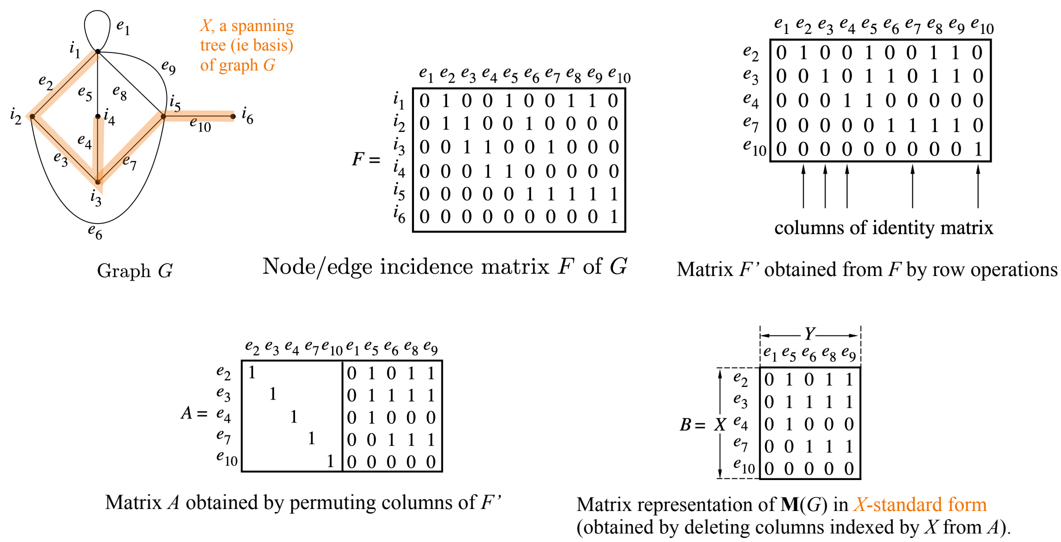

Figure 5: (*Adapted from Truemper, Matroid Decomposition*) Derivation of a standard form representation of matroid $\mathbf{M}(G)$ with respect to basis X. *Top left* an undirected multigraph G. *Top center* the node/edge incidence matrix of G, denoted F. Matrix F' obtained from F by (i) performing elementary row operations, (ii) deleting a zero row created by these operations, and (iii) relabeling row i_m by the unique element $e_p \in X$ such that column e_p has support on i_m.

Theorem 3 (**Dual standard representations**)**.** *If $B \in \mathbb{K}^{X \times Y}$ is a standard representation of* $\mathbf{M}$ *with respect to basis X, then $B^T \in \mathbb{K}^{Y \times X}$ is a Y-standard representation of X.*

Exercise 3 (Likely to take > 5 min)**.** *Prove Theorem 3.*

Hint *Fix a subset $S \subseteq E$. Persuade yourself that S is a basis of* $\mathbf{M}$ *iff $B[X - S, S - X]$ is invertible.*

10 Fundamental circuits

Theorem 4 (Existence and uniqueness of fundamental circuits)**.** *Suppose that $E(\mathbf{M}) = X \sqcup Y$, where X is a basis of* $\mathbf{M}$*. If $y \in Y$, then $X \cup \{y\}$ contains exactly one circuit $\zeta_{X,y}$.*

Definition 12 (Fundamental circuit)**.** *The circuit $\zeta_{X,y}$ defined in Theorem 4 is called the* fundamental circuit *of y with respect to X.*

Variation in meaning across fields *Graph theorists seem to use* fundamental circuits *and as* fundamental cycles *interchangeably when referring to what matroid-theorists call fundamental circuits.*

Exercise 4. *Let G and X be the graph and basis shown in Figure 5. Then the fundamental circuit of e_8 with respect to X is $\zeta_{X,y} = \{e_2, e_3, e_7, e_8\}$.*

Exercise 5. *If $E(\mathbf{M}) = X \sqcup Y$ and $B \in \mathbb{K}^{X \times Y}$ is an X-standard matrix representation of $\mathbf{M}$, then*

$$\zeta_{X,y} = \{y\} \cup \{x \in X : B[x,y] \neq 0\}$$

for each $y \in Y$.

Hint *Inspect the matrix A shown in Figure (5).*

Exercise 8. *Verify Theorem 4 for the fundamental circuits of X in Figure 5.*

Exercise 9. *Prove Theorem 4 in the special case where $\mathbf{M}$ is linearly representable.*

Exercise 10 (Challenge problem)**.** *Prove Theorem 4 in general.*

11 Duality

Definition 13 (Dual matroid)**.** *The* dual *to a matroid $\mathbf{M} = (E, \mathbf{I})$ is the matroid $\mathbf{M}^*$ such that*

$$\begin{aligned} \mathbf{B}(\mathbf{M}^*) &= \{E - X : X \in \mathbf{B}(\mathbf{M})\} \\ \mathbf{I}(\mathbf{M}^*) &= \{S : S \subseteq E - X \text{ for some } X \in \mathbf{B}(\mathbf{M})\} \\ \rho_{\mathbf{M}^*}(S) &= \max\{|S - X| : X \in \mathbf{B}(\mathbf{M})\} \end{aligned}$$

Historical note *Whitney defined the dual matroid in the same 1835 paper that introduced matroids.*

Theorem 5. *Dualization is involutive:* $(\mathbf{M}^*)^* = \mathbf{M}$.

Hyperplanes (flats of co-rank 1) enable some very elegant characterizations of dual structures (as we will see).

Definition 14 (Hyperplanes)**.** *A hyperplane is a flat of rank $\rho(E) - 1$.*

Hyperplanes may be empty, in general.

Theorem 6 (Flats are intersections of hyperplanes)**.** *Each flat F of rank $\rho(E) - r$ is an intersection of r hyperplanes.*

Proof. Let I be a maximal indpendent subset of F, and extend I to a basis $X = I \cup \{x_1, \ldots, x_n\}$ of $\mathbf{M}$. Then F is the intersection of hyperplanes $\bigcap_i \mathrm{cl}(B - \{x_i\})$. □

12 Co-objects

Definition 15 (Co-object)**.**

1. *A* cocircuit *is a circuit of* $\mathbf{M}^*$.
2. *A* coflat *is a flat of* $\mathbf{M}^*$.
3. …

In general, a coobject *is an* object *of* $\mathbf{M}^*$.

Theorem 7 (Hyperplanes are complements of cocircuits)**.** *The complement map $H \mapsto E - H$ carries hyperplanes bijectively onto cocircuits.*

Exercise 11. *Prove Theorem 7 using the definition of circuits and the dual matroid. It may be helpful to consider vector configurations as a first example.*

Theorem 8 (Flats are complements of cocircuit-unions)**.** *The complement map $Z \mapsto E - Z$ carries $\{Z : Z$ is a union of cocircuits$\}$ bijectively onto $\mathcal{F}(\mathbf{M})$.*

Exercise 12. *Prove Theorem 8, using the fact that hyperplanes are complements of cocircuits (Theorem 7) and the fact that all flats are intersections of hyperplanes (Theorem 6).*

Theorem 9 (Cocircuits are graph cuts)**.** *If $\mathbf{M}$ is the matroid of an undirected multigraph G, then the family of cocircuits of $\mathbf{M}$ is*

$$\mathbf{C}(\mathbf{M}^*) = \min \mathbf{J}$$

where $\zeta^* \in \mathbf{J}$ iff deletion of each edge in ζ^* increases the number of connected components of G by 1.

Exercise 13. *Prove Theorem 9 using the definition of circuits and the dual matroid.*

Theorem 10 (Rows of standard representations are matroid cocircuits)**.** *Suppose $B \in \mathbb{K}^{X \times Y}$ is a standard representation of $\mathbf{M}$ with respect to basis X. If $x \in X$, then $\{x\} \cup \{y \in Y : B[x,y] \neq 0\}$ is the fundamental cocircuit of x with respect to Y.*

Exercise 14. *Prove Theorem 10 by combining Theorem 3 with Exercise 5.*

Table 3: Dual properties of edge sets

	Property	A	B	C	D	E	F
complement of a basis	Cobasis		X				
complement of a hyperplane	Cocircuit	X		X		X	X
complement of a union-of-circuits	Coflat	X		X	X	X	X
	Basis	X		X	X		
	Circuit		X				
	Flat		X			X	X

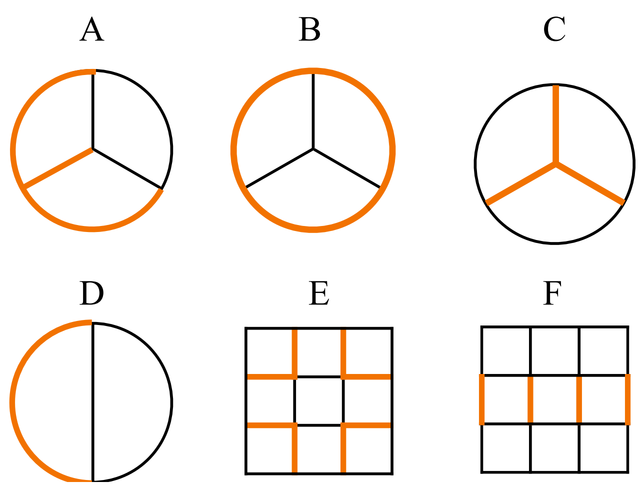

Figure 6: Edge sets for Table 3.

13 Graph minors: deletion, contraction, dualization

Deletion, contraction, and dualization are fundamental operations on an undirected planar multigraph G. They admit the following relations, for each edge e

$$G^{**} = G \qquad (G/b)^* = G^* - b \qquad (G-b)^* = G^*/b.$$

See Figure 7 for illustration.

Definition 16 (Graph minor)**.** *A* minor *of an undirected multigraph G is a graph H obtained from G by deleting edges and vertices and contracting edges.*

Variation in meaning across fields *matroid minors are similar, but exclude deletion of vertices (which has not natural matroid equivalent) and* include dualization.

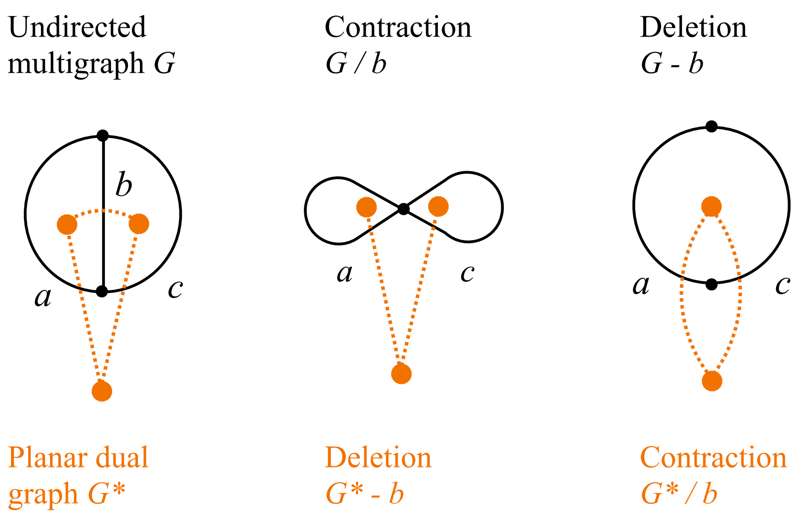

Figure 7: Graph minors: deletion, contraction, duality

Theorem 11 (Order does not matter to graph minors)**.** *Order of operations does not affect the outcome of a sequence of deletion, contraction, and dualization operations. Therefore every minor may be expressed in form*

$$((G/S)-T)^{n*}$$

for some disjoint edge sets S,T, and some $n \in \{0,1\}$.

Minors have powerful classifying properties; for example, a finite graph is planar iff it has no minor isomorphic to K_5 or $K_{3,3}$. They also feature in a wide array of powerful **graph algorithms**.

14 Matroid minors: deletion, contraction, dualization

Definition 17. *Matroid contraction The* contraction *of a matroid* $\mathbf{M} = (E, \mathbf{I})$ *by* $S \subseteq E$ *is*

$$\mathbf{M}/\ S := (E, \mathbf{J})$$

where $\mathbf{J}$ *is collection of subsets such that* $I \in \mathbf{J}$ *iff* $I \cup J \in \mathbf{I}$ *for all independent* $J \subseteq S$*.*

Theorem 1 (Contraction yields a matroid)**.** *The contraction of* $\mathbf{M}$ *by any subset S is a matroid.*

Theorem 12 (Contractions, quotients, and vector configurations)**.** *A finite dimensional vector configuration* $\phi : E \to V$ *represents* $\mathbf{M}$*, and if* $\pi : V \to V/\ \langle \phi(S) \rangle$ *is the linear quotient map, then*

$$\pi \circ \phi|_{E-S} : E - S \to V/\ \langle \phi(S)$$

represents $\mathbf{M}/\ S$*.*

Caveat lector *While contractions and quotients share many similar properties, they are fundamentally different operations. For more on the relation between these concepts, see Heunen and Patta,* The category of matroids.

Exercise 15. *Prove Theorem 12.*

Definition 18 (Deletion/restriction)**.** *If* $E = S \sqcup T$*, then the* deletion *of* $\mathbf{M}$ *by* T *(equivalently, the restriction of* $\mathbf{M}$ *to S) is the matroid*

$$\mathbf{M}|_S := \mathbf{M} - T := (S, \{I \in \mathbf{I} : I \subseteq S\}).$$

Theorem 2 (Deletion/restriction yields a matroid)**.** *The restriction of* $\mathbf{M}$ *to any subset S is a matroid.*

Definition 19 (Matroid minor)**.** *A* minor *of* $\mathbf{M}$ *is a matroid obtained by sequential deletion, contraction, and dualization operations.*

Theorem 13 (Order invariance)**.** *Order of operations does not affect the outcome of a sequence of deletion, contraction, and dualization operations. Therefore every minor may be expressed in form*

$$((\mathbf{M}/S) - T)^{n*}$$

for some disjoint edge sets S, T*, and some* $n \in \{0, 1\}$*.*

Theorem 14. *The three fundamental operations on matroids obey the same rules observed for planar graphs:*

$$\mathbf{M}^{**} = \mathbf{M} \qquad (\mathbf{M}/S)^* = \mathbf{M}^* - S \qquad (\mathbf{M} - S)^* = \mathbf{M}^*/S.$$

See Figure 7 for illustration.

15 Graphic matroids

Definition 4. *A matroid* $\mathbf{M}$ *is* graphic *if it is isomorphic to the matroid of an undirected multigraph. It is* cographic *if* $\mathbf{M}^*$ *is graphic.*

Theorem 15 (Matroid minors respect graph minors)**.** *If S and T are disjoint edge sets of a undirected planar multigraph G, then*

$$\mathbf{M}\big(((G/S) - T)^{n*}\big) = (\mathbf{M}(G)/S) - T)^{n*}$$

The same holds for $n = 0$ when G is not planar.

Definition 20 (Graphic and cographic matroids)**.** *A matroid* $\mathbf{M}$ *is* graphic *if* $\mathbf{M}$ *is isomorphic to the matroid of some combinatorial multigraph G. It is* cographic *if* $\mathbf{M}^*$ *is graphic.*

Theorem 16 (Planar $\iff$ graphic and cographic)**.** *A finite undirected multigraph is planar iff it is both graphic and cographic.*

Remark 2 (**Tests for graphicness**)**.** *There are tests for graphicness! Tutte (1960, assuming that the matroid is already known to be representable over GF(2)) and Seymour (1981, assuming only access to a matroid oracle). (*Thanks to Nick for asking*)*

16 Day 3

Recap from last time:

1. Dual matroids and co-objects.
2. Graphic matroids.
3. Deletion, restriction, contraction.

Appetizers

1. History

2. Intersection, polytopes

Takeaways:

1. The big three: duality, intersection, algorithms
2. Matroids are like old friends (you can pick back up with them any time).

17 Direct sums of matroids

Definition 5. *The* direct sum *of* $\mathbf{M}$ *and* $\mathbf{N}$ *is*

$$\mathbf{M} \oplus \mathbf{N} := (E(\mathbf{M}) \sqcup E(\mathbf{N}), \{I \sqcup J : I \in \mathbf{I}(\mathbf{M}), J \in \mathbf{I}(\mathbf{N})\})$$

18 Local history

The following are among the most classical and far-reaching results in matroid theory.

Theorem 17. *A matroid* $\mathbf{M}$ *satisfies the max-flow min-cut property iff it is binary and contains no minor isomorphic to the dual of the Fano matroid.*

University College of Swansea, 1977*:* proof by P. D. Seymour*, The matroids with the max-flow min-cut property,*

Theorem 18. *Every regular (that is, representable over every field) matroid may be constructed by piecing together graphic and cographic matroics with copies of a certain 10-element matroid.*

Merton College, Oxford, and University of Waterloo*:* proof by P. D. Seymour*, Decomposition of regular matroids,*

19 Intersection, polytopes

The matroid polytope of $\mathbf{M}$ is $P(\mathbf{M}) := \{x \in \mathbb{R}^E_{\geq 0} : \sum_{s \in S} x_s \leq \rho_{\mathbf{M}}(S) \text{ for each } S \subseteq E\}$. Equivalently, it is the convex hull of $\{\chi_I \in \mathbb{R}^E : I \in \mathbf{I}\}$. The *matroid intersection polytope* of $\mathbf{M} \cap \mathbf{N}$, denoted $P(\mathbf{M} \cap \mathbf{N})$, is defined as the convex hull of $\{\chi_I : I \in \mathbf{I}(\mathbf{M}) \cap \mathbf{I}(\mathbf{N})\}$, even though $\mathbf{I}(\mathbf{M}) \cap \mathbf{I}(\mathbf{N})$ is *not* the independence system of a matroid, in general.

Theorem 19 (Edmonds)**.**

$$P(\mathbf{M} \cap \mathbf{N}) = P(\mathbf{M}) \cap P(\mathbf{N}).$$

20 Minimal bases

Let E be a finite set and $w : E \to \mathbb{R}$ be any function. Some problems of tremendous weight can be cast in the following form: given a family of subsets $\mathbf{I} \subseteq 2^E$, find an element $I \in \mathbf{I}$ on which the *weight function*

$$w(I) := \sum_{i \in I} w_i$$

attains minimum.

Notation 1 (Flat filtration - nonstandard)**.** *We write*

$$w_{\leq t} := w^{-1}(-\infty, t] \qquad\qquad w_{=t} := w^{-1}(t).$$

Definition 21 (Flat filtration - nonstandard)**.** *The* flat filtration induced by w *is indexed family of flats* $F^w : \mathbb{R} \to \mathcal{F}(\mathbf{M})$ *such that*

$$F_t^w = \mathrm{cl}(w_{\leq t}).$$

(we will suppress the superscript when there is no chance of confusion).

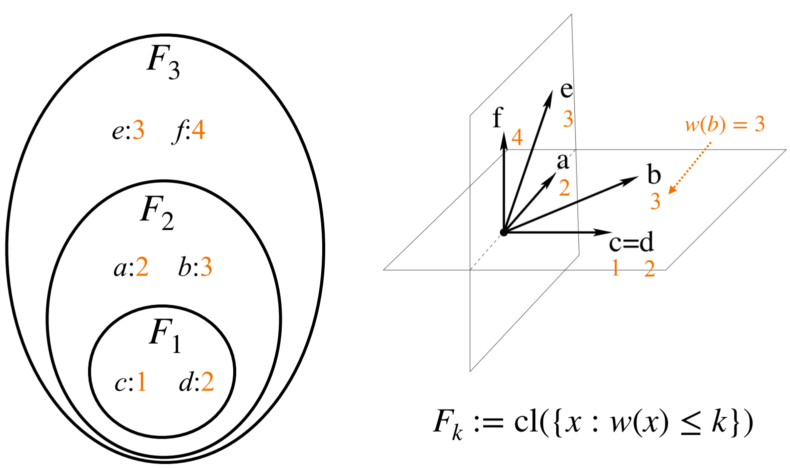

Figure 8: Proof of correctness, matroid algorithm

Exercise 6. *Suppose that* $w : E \to \mathbb{Z}_{\geq 0}$*.*

1. *If* B *is a basis, then* $|B \cap w_{\leq k}| \leq |B \cap F_k| \leq \rho(F_k)$ *for all* k*.*
2. *There exists a basis such that* $|B \cap w_{\leq k}| = \rho(F_k)$ *for all* k*.*

Theorem 20 (Matroid of min-weight bases)**.** *The following are equivalent for any basis* B*.*

1. B *is* w*-minimal.*
2. $|B \cap w_{=k}| = \rho(w_{\leq k}/\ w_{<k}) = \rho(w_{\leq k}) - \rho(w_{<k})$ *for all* k

3. $B \in \mathbf{B}\left(\bigoplus_k (w_{\leq k} / w_{<k})\right)$

In particular, the w-minimal bases define a new matroid!

Proof. See the proof of correctness for Algorithm 1. □

21 Algorithm

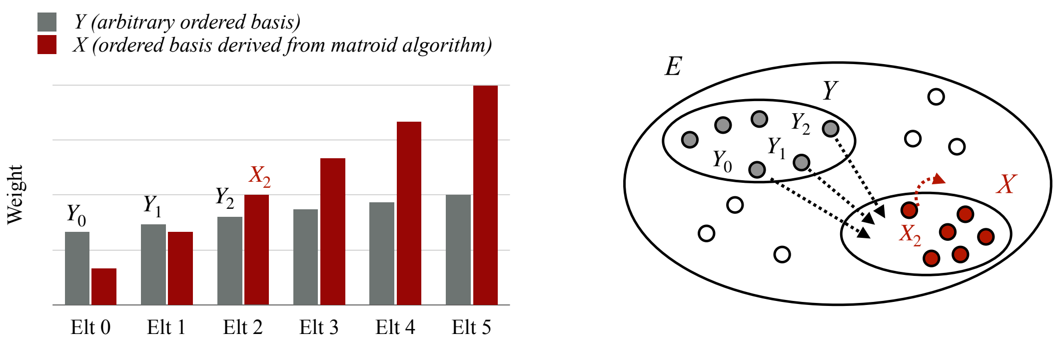

Figure 9: Proof of correctness, matroid algorithm

Notation 2. *Every finite totally ordered set* S *admits a unique poset isomorphism* $\psi : \{0 < \cdots < m\} \to \mathrm{S}$ *for some* m. *By convention, we write*

$$\mathrm{S}_p := \psi_p$$

Algorithm 1 (Matroid algorithm). *Let* $\mathrm{E} = (E, \preceq)$ *be a linear ordering of* E *such that*

$$w(E_p) < w(E_q) \implies p < q.$$

Version 1 *Return* $X := \{\mathrm{E}_p : \mathrm{E}_p \notin \mathrm{cl}(\mathrm{E}_0, \ldots, \mathrm{E}_{p-1})\}$.

Version 2 *Put* $X^{-1} := \emptyset$, *and define* X^m *recursively such that*

$$X^{m+1} = \begin{cases} X^m \cup \{\mathrm{E}_{m+1}\} & \mathrm{E}_{m+1} \notin \mathrm{cl}(\mathrm{E}_0, \ldots, \mathrm{E}_{p-1}) \\ X^m & else. \end{cases}$$

Return $X := X^{|E|-1}$.

Version 3 *Put* $Z^{-1} := \emptyset$, *and define* Z^k *recursively such that*

$$Z^{k+1} = Z^k \cup \{\min(\mathrm{E} - \mathrm{cl}(Z^k))\}$$

where minima are taken with respect to $\preceq$. *Stop when* $\mathrm{cl}(Z^k) = E$, *and return* $X := Z^k$.

Matroid characterization 8 (by algorithm). *The following are equivalent.*

1. *The matroid algorithm returns an element of* $\mathrm{argmin}_{\mathbf{I}} w_f$ *for each* $f : E \to \mathbb{R}$.
2. *Family* $\mathbf{I}$ *is the independence system of a matroid on ground set* E.

Orignial proof by Rado/Gale.

Proof. Suggestion Readers may wish to look at Figure 10 before/during their examination of the proof.

(matroid $\implies$ correctness for any w) We begin with four supporting claims.

Claim 1 Versions 1, 2, and 3 return the same set X. Proof Inspection.

Claim 2 One has $\rho(\mathrm{E}_0, \ldots, E_m) = \rho(X^m) = |X^m|$ for all m. Proof Dimension counting.

Claim 3 One has $\mathrm{cl}(\mathrm{E}_0, \ldots, \mathrm{E}_m) = \mathrm{cl}(X^m)$ for all m. Proof Dimension counting.

Claim 4 Set X is a basis. Proof Follows from Claims 2 and 3.

Now fix an arbitrary basis Y and let $\mathrm{Y} := (Y, \preceq)$ be the linear order on Y inherited from E. Suppose, for a contradiction, that $w(Y) < w(X)$. Then there must exist a p such that $w(\mathrm{Y}_p) < w(\mathrm{X}_p)$. Since $\{\mathrm{Y}_0, \ldots, \mathrm{Y}_p\}$ has one element more than $\{\mathrm{X}_0, \ldots, \mathrm{X}_{p-1}\}$ there must exist $r \leq p$ such that $\mathrm{Y}_r \notin \mathrm{cl}(\{\mathrm{X}_0, \ldots, \mathrm{X}_{p-1}\})$, that is,

$$\mathrm{Y}_r \in E - \mathrm{cl}(Z^{p-1})$$

But on the other hand $w(\mathrm{Y}_r) \leq w(\mathrm{Y}_p) < w(\mathrm{X}_p)$ implies the lefthand inequality in

$$\mathrm{Y}_r \precneqq \mathrm{X}_p = \min\{E - \mathrm{cl}(Z^{p-1})\}$$

a contradiction. The desired conclusion follows.

(correctness for any $w \implies$ matroid) See Oxley, *Matroid theory*. $\square$

Remark 3. *Remarkably, this discussion shows that optimal basis "don't care" about magnitude, only order. More precisely, the optimal bases of* w *and* $f \circ w$ *are identical, whenever* $f : \mathbb{R} \to \mathbb{R}$ *is strictly increasing.*

21.1 Special topic: graphs, greedoids, spanning trees

Special thanks to Nicholas Sale for a nice summary!

A min-weight basis for the matroid of a (connected) undirected multigraph is called a min-weight spanning trees. There are many theorems/algorithms about spanning trees in graph theory literature.

Kurskal's algorithm coincides exactly with the matroid algorithm. One builds a sequence of independent sets $I_0 \subseteq \cdots I_n$ by added edges of minimal weight. Critically, I_k may be a forest and not a tree for $0 < k < n$.

Prim's algorithm Prim's algorithm likewise builds a sequence of independent sets $I_0 \subseteq \cdots I_n$, but guarantees that each is a tree. The family $\mathbf{J}$ of trees of a graph is not a matroid in general, since one can disconnect a tree by removing an edge. This leads naturally to the notion of a *greedoid*, which is a relaxation of the notion of a matroid.

Definition 22 (greedoid)**.** *A greedoid on a finite set E is a pair $(E, \mathbf{F})$ where $\mathbf{F}$ is a family of subsets of E such that*

1. $\emptyset \in \mathbf{F}$
2. $S \in \mathbf{F} - \{\emptyset\} \implies \exists s \in S$ *such that* $S - \{s\} \in \mathbf{F}$
3. $S, T \in \mathbf{F} \wedge |S| = |T| + 1 \implies \exists s \in S$ *such that* $T \cup \{s\} \in \mathbf{F}$.

Greedoids have many attractive computational properties, just like matroids!

22 Application: homological cycle bases

Theorem 21. *If Z_p is the space of p-cycles of a $\mathbb{K}$-linear chain complex C, and if $\mathbf{I}_p$ is the set of all $S \subseteq Z_p$ such that $\{[s] \in H_p(C) : s \in S\}$ spans a subspace of dimension $|S|$, then $\mathbf{M}_p := (Z_p, \mathbf{I}_p)$ is a matroid.*

Proof. Matroid $\mathbf{M}_p$ is equal to the contraction minor $Z_p / \partial(C_{p+1})$. □

Definition 6. *We call $\mathbf{B}_p := \mathbf{B}(\mathbf{M}_p)$ the family of p-dimensional* homology cycle bases *of C.*

Corollary 1. *Min-weight homology cycle bases can be calculated by the matroid algorithm!*

History and application *This observation appeared in Erickson, Jeff, and Kim Whittlesey.* Greedy optimal homotopy and homology generators. *SODA. Vol. 5. 2005. Despite being quite recent, much work has been gotten from it (244 citations on last count).*

Remark 4. *Naive application of the matroid algorithm requires exhaustive search over every vector in the cycle space Z_p, hence works only over finite fields, and is not very appealing in practice. There are work-arounds, however.*

Remark 5. *For the many applications, it makes more sense to speak of a minimal* basis *of cycle representatives than a minimal representative.*

Exercise 7. *True or False? Every finite filtered CW complex has a homological cycle basis* composed exclusively of matroid circuits*? (Hint: True)*

Exercise 8 (Preview - we'll return to this when we have more tools)**.** *True or False? Every finite filtered CW complex has a persistent homology cycle basis* composed exclusively of matroid circuits*? (Hint: False)*

23 Bioptimal bases

Definition 7. *A basis is (w_0, w_1)-optimal, or* bioptimal*, if it is optimal with respect to both w_0 and w_1.*

Notation 1. *By abuse of notation, we will identify a filtration $\mathbf{F}_0 \subseteq \cdots \subseteq \mathbf{F}_n = E$ of a set E with the function $\chi_{\mathbf{F}} : E \to \mathbb{R}$ such that $\mathbf{F}_i = \chi_{\mathbf{F}}^{-1}(-\infty, i]$ for all i.*

Exercise 9. *Consider the following (false) assertion: given weight functions $w_0, w_1 : E \to \mathbb{Z}_{\geq 0}$ there exists a basis B which is both w_0-minimal and w_1-minimal.*

1. *Find the error in the following "proof" The w_0-optimal bases define a matroid $\mathbf{N}$ on E. Apply the matroid algorithm to obtain a w_1-minimal basis B of $\mathbf{N}$. It is also w_0-optimal,* a fortiori.
2. *Confirm that the following counterexample proves the theorem false: $E(\mathbf{M}) := \{e_0, e_1\}$, $\mathbf{B}(\mathbf{M}) := \{\{e_0\}, \{e_1\}\}$, and $w_i(e_j) = |i - j|$.*

Theorem 22 (H.-P., Ghrist)**.** *Suppose that $\mathbf{F}_0 \subseteq \cdots \subseteq \mathbf{F}_m = E$ and $\mathbf{G}_0 \subseteq \cdots \subseteq \mathbf{G}_n = E$ are two filtrations of $\mathbf{M}$, and $\mathbf{F}_i, \mathbf{G}_i \in \mathcal{F}(\mathbf{M})$ for all i. Then the following are equivalent*

1. *There exists an $(\mathbf{F}, \mathbf{G})$-bioptimal basis.*
2. *One has $\rho(\mathbf{F}_i) + \rho(\mathbf{G}_j) = \rho(\mathbf{F}_i \cup \mathbf{G}_j) + \rho(\mathbf{F}_i \cap \mathbf{G}_j)$ for all i and j.*
3. *One has*

$$\rho(\mathbf{M}) = \sum_{i,j} \rho\left(\frac{\mathbf{F}_i \cap \mathbf{G}_j}{(\mathbf{F}_{i-1} \cup \mathbf{G}_{j-1}) \cap \mathbf{F}_i \cap \mathbf{G}_j}\right)$$

for all i and j.

In this case the bioptimal bases coincide exactly with the bases of

$$\bigoplus_{i,j} \left(\frac{\mathbf{F}_i \cap \mathbf{G}_j}{(\mathbf{F}_{i-1} \cup \mathbf{G}_{j-1}) \cap \mathbf{F}_i \cap \mathbf{G}_j}\right)$$

Proof and discussion *See H.-P. and Ghrist,* Matroid filtrations and computational persistent homology*, 2017*

Corollary 2. *If $\mathbf{F}$ and $\mathbf{G}$ are two filtrations of a vector space V by linear subspaces, then there exists an $(\mathbf{F}, \mathbf{G})$-minimal basis.*

Remark 2. *Typically the intersection of two matroids does not yield a matroid. The preceding theorem therefore says something very strong about the structural properties of pairs of filtrations.*

24 Linear filtrations and matrix reduction

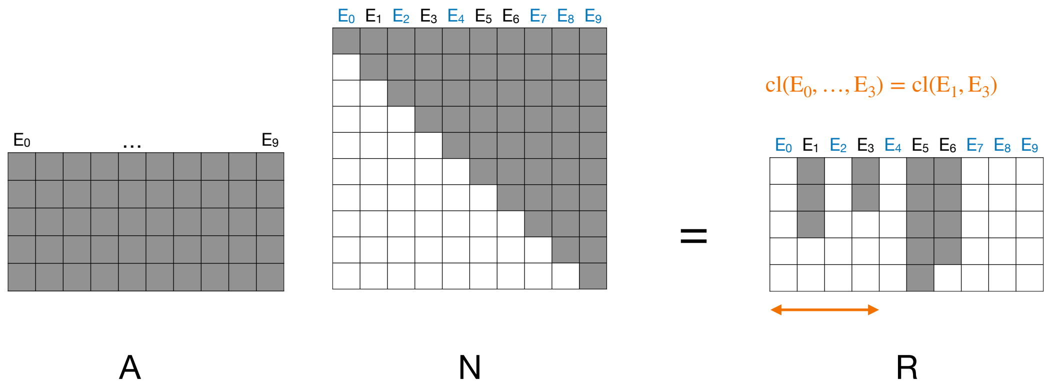

Figure 10: Calculation of optimal bases for image and kernel filtrations.

Definition 23 (low, reduced)**.** *Given a matrix $A \in \mathbb{K}^{F \times E}$ with totally ordered rows,* low^A*, or simply* low *is the partially defined function on E such that $\mathrm{low}_e = \max\{f \in F : A[f,e] \neq 0\}$ when $A[:,e]$ is nonzero, and low_e is undefined otherwise. We say that A is* reduced *if low^A is injective on its domain of definition.*

25 Persistent cycle bases

Theorem 23. *Let $C = (C^i)_{i<m}$ be filtered chain complex and Z_n the space of n-cycles of C. Let $\mathbf{F} = (\mathbf{F}_0 \subseteq \cdots \subseteq \mathbf{F}_m = Z_m)$ and $\mathbf{G} = (\mathbf{G}_0 \subseteq \cdots \subseteq \mathbf{G}_m = Z_m)$ be the filtrations on Z_n induced by intersection with C^i_n and direct image of C^i_{n+1}, respectively (that is, $\mathbf{F}$ is filtration by birth, $\mathbf{G}$ by death). Then the persistent cycle bases are exactly*

$$\bigoplus_{i<j} \left(\frac{\mathbf{F}_i \cap \mathbf{G}_j}{(\mathbf{F}_{i-1} \cup \mathbf{G}_{j-1}) \cap \mathbf{F}_i \cap \mathbf{G}_j} \right)$$

Exercise 10. *True or False? Every finite filtered CW complex has a persistent homology cycle basis* composed exclusively of matroid circuits*? (Hint: False)*